Electronic Companion for

"Conveying Demand Information in Serial Supply Chains with Capacity Limits"

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A Proofs

Proof of Lemma 5 We first show by induction that

$$\tilde{a}_t^1 = d_{t-1}(K) \text{ and } \tilde{x}_t^1 = \tilde{z}^1 - \tilde{B}_t^{12} - (D_{t-1} - D_{t-2}(K)). \tag{A1}$$

Consider period 1. $\tilde{a}_1^1 = (\tilde{z}^1 - \tilde{x}_1^1 - \tilde{B}_1^{12}) \wedge K = (\tilde{z}^1 - \tilde{z}^1 - 0) \wedge K = 0$, $\tilde{x}_2^1 = \tilde{x}_1^1 + \tilde{q}_1^{12} - d_1 = \tilde{z}^1 - d_1$, and $\tilde{a}_2^1 = (\tilde{z}^1 - \tilde{x}_2^1 - \tilde{B}_2^{12}) \wedge K = (\tilde{z}^1 - \tilde{z}^1 + d_1) \wedge K = d_1 \wedge K = d_1(K)$. Assume these hold for t. Now

$$\begin{aligned}
\tilde{x}_{t+1}^1 &= \tilde{x}_t^1 + \tilde{q}_t^{12} - d_t \\
&= \tilde{z}^1 - \tilde{B}_t^{12} - (D_{t-1} - \tilde{D}_{t-2}) + \tilde{q}_t^{12} - d_t \\
&= \tilde{z}^1 - \tilde{B}_{t+1}^{12} + \tilde{a}_t^1 - D_t + D_{t-2}(K) \\
&= \tilde{z}^1 - \tilde{B}_{t+1}^{12} - (D_t - D_{t-1}(K))
\end{aligned} \tag{A2}$$

where the first and third equalities arises from the induction assumptions and the second arises from the $\tilde{B}_{t+1}^{12}$ transition function. Now,

$$\begin{aligned}
\tilde{a}_{t+1}^1 &= [\tilde{z}^1 - (\tilde{x}_{t+1}^1 + \tilde{B}_{t+1}^{12})] \wedge K \\
&= [D_t - D_{t-1}(K)] \wedge K \\
&= d_t(K)
\end{aligned}$$

where the first equality arises from the induction assumption. Notice from Lemma 1 that $\vec{a}_t^1 = d_{t-1}(K)$ which demonstrates the remaining equality. ■

Proof of Lemma 6 Lemma 5 shows that $SS(K) \oplus d \overset{xR}{\equiv} SS(\infty) \oplus d(K)$. Clearly, $(d(K))(K) = d(K)$, thus $SS(K) \oplus d(K) \equiv SS(\infty) \oplus d(K)$. ■

Proof of Lemma 7 From Lemma 5, $\tilde{a}_t^1 = d_{t-1}(K)$. Thus, $\tilde{a}_t^1 = \tilde{z}^1 - \tilde{x}_t^1 - \tilde{B}_t^{12}$. Summing the orders until installation i is simply, $\tilde{a}_t^1 + \tilde{a}_t^2 + ... + \tilde{a}_t^i = \sum_{k \le i} \tilde{a}_t^k = \tilde{z}^1 - \tilde{x}_t^1 - \tilde{B}_t^{12} + \tilde{z}^2 - \tilde{x}_t^2 + \tilde{B}_t^{12} - \tilde{B}_t^{23} + ... + \tilde{z}^i - \tilde{x}_t^i + \tilde{B}_t^{i-1,i} - \tilde{B}_t^{i,i+1} = \sum_{k \le i} \tilde{z}^k - \sum_{k \le i} \tilde{x}_t^k - \tilde{B}_t^{i,i+1}$. The special case of $i = N$ results in $\sum_{k \le N} \tilde{a}_t^k = \sum_{k \le N} \tilde{z}^k - \sum_{k \le N} \tilde{x}_t^k$. ■

Proof of Lemma 8 Consider a general installation $i + 1$:

$$\begin{aligned}
\tilde{a}_{t+2}^{i+2} &= \tilde{z}^{i+2} - \tilde{x}_{t+2}^{i+2} + \tilde{B}_{t+2}^{i+1,i+2} - \tilde{B}_{t+2}^{i+2,i+3} \\
&= \tilde{z}^{i+2} - \tilde{x}_{t+1}^{i+2} - \tilde{q}_{t+1}^{i+2,i+3} + \tilde{q}_{t+1}^{i+1,i+2} + \tilde{B}_{t+1}^{i+1,i+2} + \tilde{a}_{t+1}^{i+1} - \tilde{q}_{t+1}^{i+1,i+2} - \tilde{B}_{t+1}^{i+2,i+3} - \tilde{a}_{t+1}^{i+2} + \tilde{q}_{t+1}^{i+2,i+3} \\
&= \tilde{a}_{t+1}^{i+1}
\end{aligned}$$

where the first equality arises from equation (14) and the second one from equations (2) and (5).■

Proof of Theorem 2 First we prove that statements $a(t, i)$, $b(t, i)$, and $c(t, i)$ hold for $i = N$. Then, we follow downward induction for $i = N - 1, \ldots, 2$.

Assume $a(t,N)$ holds for $t' < t$, $b(t,N)$ holds for $t' < t-1$, and $c(t,N)$ holds for $t' < t-2$. From Propositions 2 and 3, $\tilde{x}^N_{t-1} + d_{t-N-1} - \tilde{B}^{N-1,N}_{t-1} = \tilde{z}^N$ and $\vec{x}^{2N-2}_{t-N} + \vec{x}^{2N-1}_{t-N} - \vec{B}^{2N-3,2N-2}_{t-N} = \vec{z}^{2N-2} + \vec{z}^{2N-1}$. Since, by design, $\tilde{z}^N = \vec{z}^{2N-2} + \vec{z}^{2N-1}$, we have:

$$
\begin{aligned}
\tilde{x}^N_{t-1} + d_{t-N-1} - \tilde{B}^{N-1,N}_{t-1} &= \vec{x}^{2N-2}_{t-N} + \vec{x}^{2N-1}_{t-N} - \vec{B}^{2N-3,2N-2}_{t-N} \\
-\vec{q}^{2N-3,2N-2}_{t-N} + d_{t-N-1} - \tilde{B}^{N-1,N}_{t-1} &= -\vec{B}^{2N-3,2N-2}_{t-N} \\
-\vec{q}^{2N-3,2N-2}_{t-N} + \vec{a}^{2N-3}_{t-N} - \tilde{B}^{N-1,N}_{t-1} &= -\vec{B}^{2N-3,2N-2}_{t-N} \\
\vec{B}^{2N-3,2N-2}_{t-N+1} = \vec{B}^{2N-3,2N-2}_{t-N} + \vec{a}^{2N-3}_{t-N} - \vec{q}^{2N-3,2N-2}_{t-N} &= \tilde{B}^{N-1,N}_{t-1}
\end{aligned}
$$

which proves $b(t-1,N)$. Using this,

$$\vec{B}^{2N-3,2N-2}_{t-N} + \vec{a}^{2N-3}_{t-N} - \vec{q}^{2N-3,2N-2}_{t-N} = \tilde{B}^{N-1,N}_{t-1} = \tilde{B}^{N-1,N}_{t-2} + \tilde{a}^{N-1}_{t-2} - \tilde{q}^{N-1,N}_{t-2}.$$

Since $\vec{B}^{2N-3,2N-2}_{t-N} = \tilde{B}^{N-1,N}_{t-2}$ (by the induction assumption $b(t-2,N)$) and $\vec{a}^{2N-3}_{t-N} = \tilde{a}^{N-1}_{t-2}$ (by Lemma 8), we get $c(t-2,N)$: $\vec{q}^{2N-3,2N-2}_{t-N} = \tilde{q}^{N-1,N}_{t-2}$. Now applying

$$\tilde{x}^N_t = \tilde{x}^N_{t-1} + \tilde{a}^N_{t-1} - \tilde{q}^{N-1,N}_{t-1}$$

We get:

$$
\begin{aligned}
\vec{x}^{2N-2}_{t-N+1} &= \vec{x}^{2N-2}_{t-N} + \vec{q}^{2N-2,2N-1}_{t-N} - \vec{q}^{2N-3,2N-2}_{t-N} \\
\vec{x}^{2N-1}_{t-N+1} &= \vec{x}^{2N-1}_{t-N} + \vec{q}^{2N-1,2N}_{t-N} - \vec{q}^{2N-2,2N-1}_{t-N}
\end{aligned}
$$

Summing,

$$
\begin{aligned}
\vec{x}^{2N-2}_{t-N+1} + \vec{x}^{2N-1}_{t-N+1} &= \vec{x}^{2N-2}_{t-N} - \vec{q}^{2N-3,2N-2}_{t-N} + \vec{x}^{2N-1}_{t-N} + \vec{a}^{2N-1}_{t-N} \\
&= \tilde{x}^N_{t-1} + \vec{a}^{2N-1}_{t-N} \\
&= \tilde{x}^N_t + \vec{q}^{2N-3,2N-2}_{t-N+1},
\end{aligned}
$$

since $\vec{a}^{2N-1}_{t-N} = \tilde{a}^N_{t-1} = \tilde{q}^{N-1,N}_{t-1} = \vec{q}^{2N-3,2N-2}_{t-N+1}$ (we use here $b(t-1,N)$). This proves $a(t,N)$. The initial step for starting t is easy to verify. Thus this completes the proof for all t and $i = N$.

Now consider installation $i = 2,...,N-1$ and assume $a(t,i)$, $b(t,i)$ and $c(t,i)$ hold for all t and $i' > i$. Specifically, we have $b(t-1,i+1)$: $\tilde{B}^{i,i+1}_{t-1} = \vec{B}^{2i-1,2i}_{t-i}$ and $c(t,i+1)$: $\tilde{q}^{i,i+1}_t = \vec{q}^{2i-1,2i}_{t-i+2}$ for all t. As $\tilde{z}^i = \vec{z}^{2i-2} + \vec{z}^{2i-1}$, from Propositions 2 and 3:

$$
\begin{aligned}
\tilde{x}^i_{t-1} + d_{t-i-1} - \tilde{B}^{i-1,i}_{t-1} + \tilde{B}^{i,i+1}_{t-1} &= \vec{x}^{2i-2}_{t-i} + \vec{x}^{2i-1}_{t-i} - \vec{B}^{2i-3,2i-2}_{t-i} + \vec{B}^{2i-1,2i}_{t-i} \\
-\vec{q}^{2i-3,2i-2}_{t-i} + d_{t-i-1} - \tilde{B}^{i-1,i}_{t-1} &= -\vec{B}^{2i-3,2i-2}_{t-i} \\
-\vec{q}^{2i-3,2i-2}_{t-i} + \vec{a}^{2i-3}_{t-i} - \tilde{B}^{i-1,i}_{t-1} &= -\vec{B}^{2i-3,2i-2}_{t-i} \\
\vec{B}^{2i-3,2i-2}_{t-i+1} = \vec{B}^{2i-3,2i-2}_{t-i} + \vec{a}^{2i-3}_{t-i} - \vec{q}^{2i-3,2i-2}_{t-i} &= \tilde{B}^{i-1,i}_{t-1}.
\end{aligned}
$$

Thus $b(t,i)$ holds for all t. Using $b(t-1,i)$,

$$[\vec{B}^{2i-3,2i-2}_{t-i+1} =]\vec{B}^{2i-3,2i-2}_{t-i} + \vec{a}^{2i-3}_{t-i} - \vec{q}^{2i-3,2i-2}_{t-i}[= \tilde{B}^{i-1,i}_{t-1}] = \tilde{B}^{i-1,i}_{t-2} + \tilde{a}^{i-1}_{t-2} - \tilde{q}^{i-1,i}_{t-2}.$$

Since from $b(t-2,i)$: $\vec{B}^{2i-3,2i-2}_{t-i} = \tilde{B}^{i-1,i}_{t-2}$ and (by Lemma 8) $\vec{a}^{2i-3}_{t-i} = \tilde{a}^{i-1}_{t-2}$, we get $c(t-2,i)$: $\vec{q}^{2i-3,2i-2}_{t-i} = \tilde{q}^{i-1,i}_{t-2}$. Thus $c(t,i)$ holds for all t. The remaining claim $a(t,i)$ is proved using forward induction on t. That is, we assume $a(t-1,i)$ holds. Summing,

$$\vec{x}^{2i-2}_{t-i+1} = \vec{x}^{2i-2}_{t-i} + \vec{q}^{2i-2,2i-1}_{t-i} - \vec{q}^{2i-3,2i-2}_{t-i} \text{ and} \tag{A3}$$

$$\vec{x}^{2i-1}_{t-i+1} = \vec{x}^{2i-1}_{t-i} + \vec{q}^{2i-1,2i}_{t-i} - \vec{q}^{2i-2,2i-1}_{t-i}, \tag{A4}$$

we have

$$\begin{aligned}
\vec{x}^{2i-2}_{t-i+1} + \vec{x}^{2i-1}_{t-i+1} &= \vec{x}^{2i-2}_{t-i} - \vec{q}^{2i-3,2i-2}_{t-i} + \vec{x}^{2i-1}_{t-i} + \vec{q}^{2i-1,2i}_{t-i} \\
&= \tilde{x}^{i}_{t-1} + \vec{q}^{2i-1,2i}_{t-i} \qquad \text{using } a(t-1,i) \\
&= \tilde{x}^{i}_{t-1} + \tilde{q}^{i,i+1}_{t-1} \qquad \text{using } c(t-1,i+1) \\
&= \tilde{x}^{i}_{t} + \tilde{q}^{i-1,i}_{t-1} \\
&= \tilde{x}^{i}_{t} + \vec{q}^{2i-3,2i-2}_{t-i+1}.
\end{aligned}$$

Thus, $a(t,i)$ holds, which proves $a(t,i)$ for all t. ■

Proof of Proposition 4 Statement (i) follows from Theorem 2 $a(t,N)$, equation (2), and $\vec{q}^{2N-1,2N}_{t-N+1} = d_{t-N}(K)$. For statement (ii), consider the following two propositions:

$$\text{P1(N)} \qquad \sum_{i=1}^{N} \tilde{x}^i_t = \vec{Z}^{2N-1} - \sum_{i=1}^{N} d_{t-i}(K)$$

and

$$\text{P2(N)} \qquad \sum_{i=1}^{2N-1} \vec{x}^i_t = \vec{Z}^{2N-1} - d_{t-1}(K).$$

(P2(N) can be proven with a simple induction of the summation of the inventory levels in *SF*.) To prove P1(N), we sum the transition functions of $\tilde{x}^1_t$ and $\tilde{x}^i_t$ for $i > 1$, resulting in

$$\sum_{i=1}^{N} \tilde{x}^i_t = \sum_{i=1}^{N} \tilde{x}^i_{t-1} - d_{t-1}(K) + \tilde{a}^N_{t-1} = \sum_{i=1}^{N} \tilde{x}^i_{t-1} - d_{t-1}(K) + d_{t-N-1}(K).$$

Assume

$$\begin{aligned}
\sum_{i=1}^{N} \tilde{x}^i_{t-1} &= \vec{Z}^{2N-1} - \sum_{i=1}^{N} d_{t-i-1}(K) \\
\sum_{i=1}^{N} \tilde{x}^i_t + d_{t-1}(K) - d_{t-N-1}(K) &= \vec{Z}^{2N-1} - \sum_{i=1}^{N} d_{t-i-1}(K) \\
\sum_{i=1}^{N} \tilde{x}^i_t &= \vec{Z}^{2N-1} - \sum_{i=1}^{N} d_{t-i}(K)
\end{aligned}$$

Inserting $\vec{Z}^{2N-1} = \sum_{i=1}^{N} \tilde{x}_t^i + \sum_{i=1}^{N} d_{t-i}(K)$ into P2(N) gives

$$\begin{aligned} \sum_{i=1}^{N} \tilde{x}_t^i + \sum_{i=1}^{N} d_{t-i}(K) - d_{t-1}(K) &= \sum_{i=1}^{2N-1} \vec{x}_t^i \\ \sum_{i=2}^{N} \tilde{x}_t^i + \sum_{i=2}^{N} d_{t-i}(K) &= \sum_{i=2}^{2N-1} \vec{x}_t^i \end{aligned}$$

.

Statement (iii) follows inductively from Theorem 2 $c(t,2)$, since the inventories, demands, and shipments to the retailer will be identical. ■

Proof of Lemma 9 The logic follows that of Lemma 1 in Parker and Kapuściński (2004) which states that for any sample path, any policy such that $y^j > \max(K_1, x^j - a^{j-1})$ for $j = 2, ..., n-1$ or $y^j > \max(K_n, x^j - a^{j-1})$ for $j = n+1, ..., N$ will be more expensive than one where $y^j \leq \max(K_1, x^j - a^{j-1})$ for $j = 2, ..., n-1$ or $y^j \leq \max(K_n, x^j - a^{j-1})$ for $j = n+1, ..., N$. The logic behind this is simply that the system gains no benefit from stocking more than the respective capacity level since the immediate downstream installation will never order more than that amount and thus only greater holding costs would be incurred if $x^j > K_1$ for $j = 2, ..., n-1$ or $x^j > K_n$ for $j = n+1, ..., N$. ■

Proof of Lemma 10 Consider two systems, A and B. System A operates under *2-MEBS*(Z, K_1, K_n) and system B operates under *2-MEBS*$(Z(K), K_1, K_n)$. Note that $Z^i(K) = Z^{i-1}(K) + (Z^i - Z^{i-1}(K)) \wedge K$ for $i = 2, ..., n-1$. Consider a specific period (index omitted) and then installations 1 to N. For installation 1, $Z^1 = Z^1(K)$ and so $Y^{1A} = Y^{1B}$ since both systems start from the same inventory vector. Whenever the inventory availability limit binds, the ordering decision will be common between the systems so we will omit this case for conciseness. Assume $Y^{i-1,A} = Y^{i-1,B}$ for a specific i for $i = 2, ...n-2, n, ..., N-1$.

For installation i, if $X^i < Z^{i-1}(K)$ then $Y^{iA} = Y^{iB} = Y^{i-1} + K$. If $X^i \geq Z^{i-1}(K)$, $Y^{iA} = Y^{iB} = Z^i(K)$ and we have two cases (i) $Z^i - Z^{i-1}(K) < K$ resulting in $Y^{iA} = \min(Z^i, X^{i+1}) = \min(Z^i(K), X^{i+1}) = Y^{iB}$, and (ii) $Z^i - Z^{i-1}(K) \geq K$ resulting in $Y^{iA} = \min(Z^{i-1}(K) + K, X^{i+1})$. Thus, $Y^{iA} = \min(\min(Z^i, Z^{i-1}(K) + K), X^{i+1}) = \min(Z^{i-1}(K) + \min(Z^i - Z^{i-1}(K), K), X^{i+1}) = \min(Z^i(K), X^{i+1}) = Y^{iB}$. For installation n, since $Z^n = Z^n(K)$ and the initial inventory vector is identical between systems A and B, $Y^{nA} = Y^{nB}$. The ordering decisions will be the same in systems A and B at all echelons, so each system will begin with identical inventory vectors in the following period. ■

Proof of Lemma 11 For $SF(K_1, K_n)$ with installation target levels z^i, when Condition I holds:

(a) $\vec{a}_t^1 = d_{t-1}(K_1)$;

(b) $\vec{a}_t^{i-1} = \vec{a}_t^i$ for $i = 2, ..., n-1, n+1, ...N$ and $\vec{a}_t^n = a_t^n(K_n)$; and

(c) $\vec{q}_t^{i,i+1} \leq K_1$ for $i = 2, ..., n-1$ and $\vec{q}_t^{i,i+1} \leq K_n$ for $i = n, ..., N-1$.

For (a), we gain $\vec{a}_t^1 = d_{t-1}(K_1)$ from Lemma 1 so the retailer's orders in $SF(K_1, K_n) \oplus d$ are the same as $SF(\infty, \infty) \oplus d(K_1, K_n)$. For (b), assume $\vec{x}_t^i = \vec{z}^i - \vec{B}_t^{i,i+1} + \vec{B}_t^{i-1,i}$ for $i = 2, ..., j-1, j+1, ..., N$. Installation 1 places an order $\vec{a}_t^1 = [\vec{z}^1 - (\vec{x}_t^1 + \vec{B}_t^{12})] \wedge K_1 \leq K_1$. Installation i orders $\vec{a}_t^i = \vec{z}^i - (\vec{x}_t^i + \vec{B}_t^{i,i+1}) + (\vec{a}_t^{i-1} + \vec{B}_t^{i-1,i}) = \vec{z}^i - (\vec{z}^i - \vec{B}_t^{i,i+1} + \vec{B}_t^{i-1,i} + \vec{B}_t^{i,i+1}) + (\vec{a}_t^{i-1} + \vec{B}_t^{i-1,i}) = \vec{a}_t^{i-1}$. Now consider the inventory transition function for installation $i > 1$, $\vec{x}_{t+1}^i = \vec{x}_t^i + \vec{q}_t^{i,i+1} - \vec{q}_t^{i-1,i}$. Add $\vec{B}_{t+1}^{i,i+1} - \vec{B}_{t+1}^{i-1,i}$ to both sides: $\vec{x}_{t+1}^i + \vec{B}_{t+1}^{i,i+1} - \vec{B}_{t+1}^{i-1,i} = \vec{x}_t^i + \vec{q}_t^{i,i+1} - \vec{q}_t^{i-1,i} + \vec{B}_{t+1}^{i,i+1} - \vec{B}_{t+1}^{i-1,i}$. Since, $\vec{B}_{t+1}^{i,i+1} + \vec{q}_t^{i,i+1} = \vec{a}_t^i + \vec{B}_t^{i,i+1}$, using the induction assumption, $\vec{x}_t^i = \vec{z}^i - \vec{B}_t^{i,i+1} + \vec{B}_t^{i-1,i}$, we have $\vec{x}_{t+1}^i + \vec{B}_{t+1}^{i,i+1} - \vec{B}_{t+1}^{i-1,i} = \vec{x}_t^i + \vec{a}_t^i + \vec{B}_t^{i,i+1} - \vec{a}_t^{i-1} - \vec{B}_t^{i-1,i} = \vec{z}^i + \vec{a}_t^i - \vec{a}_t^{i-1} = \vec{z}^i$. The induction assumption holds trivially for $t = 1$. For installation n, assume $\vec{a}_{t-1}^n = a_{t-1}^n(K_n)$

$$
\begin{aligned}
\vec{a}_t^n &= [z^n - (x_t^n - (a_t^{n-1} + B_t^{n-1,n}) + B_t^{n,n+1})] \wedge K_n \\
&= [z^n - x_1^n - \sum_{i=1}^{t-1}(q_i^{n,n+1} - q_i^{n-1,n}) + a_t^{n-1} + B_1^{n-1,n} \\
&\quad + \sum_{i=1}^{t-1}(a_i^{n-1} - q_i^{n-1,n}) - B_1^{n,n+1} - \sum_{i=1}^{t-1}(a_i^n - q_i^{n,n+1})] \wedge K_n \\
&= [\sum_{i=1}^{t} a_i^{n-1} - \sum_{i=1}^{t-1} a_i^n] \wedge K_n \\
&= [A_t^{n-1} - A_{t-1}^n(K_n)] \wedge K_n = a_t^n(K_n)
\end{aligned}
$$

Since the retailer limits her order quantity to be no greater than K_1, this property is inherited by each higher installation, $\vec{a}_t^i = \vec{a}_t^1 \leq K_1$ for all $i = 2, ..., n-1$. Similarly, $\vec{a}_t^i = \vec{a}_t^n = a_t^n(K_n) \leq K_n$ for all $i = n+1, ..., N$.

The proof of (c) is by induction on the echelon, starting from the uppermost installation. For installation N, $\vec{B}_t^{N,N+1} = 0$ (since the outside supplier has an unlimited supply, $\vec{x}_t^{N+1} = \infty$) and $\vec{q}_t^{N,N+1} = (\vec{a}_t^N + \vec{B}_t^{N,N+1}) \wedge \vec{x}_t^{N+1} = \vec{a}_t^N \leq K_n$. Assume $\vec{q}_t^{i+1,i+2} \leq K_n$ for $i = n+1, ...N-1$. Then we have two cases: (b1) $\vec{B}_t^{i,i+1} = 0$, and (b2) $\vec{B}_t^{i,i+1} > 0$. Under case (b1), $\vec{q}_t^{i,i+1} = (\vec{a}_t^i + \vec{B}_t^{i,i+1}) \wedge \vec{x}_t^{i+1} \leq \vec{a}_t^i \leq K_n$. Under case (b2), we wish to show that if there is a backorder at installation $i+1$ (owed to installation i), installation $i+1$ exhausted its supply, prior to receiving its own shipment, in the previous period. Consider $\vec{B}_t^{i,i+1} = \vec{B}_{t-1}^{i,i+1} + \vec{a}_{t-1}^i - \vec{q}_{t-1}^{i,i+1} > 0$. $\vec{B}_t^{i,i+1} > 0$ if $\vec{q}_{t-1}^{i,i+1} < \vec{B}_{t-1}^{i,i+1} + \vec{a}_{t-1}^i$ and therefore $\vec{q}_{t-1}^{i,i+1} = \vec{x}_{t-1}^{i+1}$. Now, $\vec{x}_t^{i+1} = \vec{x}_{t-1}^{i+1} + \vec{q}_{t-1}^{i+1,i+2} - \vec{q}_{t-1}^{i,i+1} = \vec{x}_{t-1}^{i+1} + \vec{q}_{t-1}^{i+1,i+2} - \vec{x}_{t-1}^{i+1} = \vec{q}_{t-1}^{i+1,i+2} \leq K_n$. Since installation $i+1$ has no more than K_n units in period t, he cannot ship more than K_n units. Identical logic applies to installation $i = 2, ..., n$.

The retailer and installation n's orders in $SF(K_1, K_n) \oplus d$ are the same as in $SF(\infty, \infty) \oplus d(K_1, K_n)$. Since all higher stages (installations $i \geq 2$) are identical in $SF(\infty, \infty)$ and $SF(K_1, K_n)$ systems, we have $SF(K_1, K_n) \oplus d \overset{xR}{\equiv} SF(\infty, \infty) \oplus d(K_1, K_n)$. ■

Proof of Lemma 12 For installations below stage n, the earlier single-band results apply, so we turn our attention to installation n.

$$
\vec{a}_t^n = [z^n - [\vec{x}_t^n - (\vec{a}_t^{n-1} + \vec{B}_t^{n-1,n}) + \vec{B}_t^{n,n+1}]] \wedge K_n
$$

$$= \quad [\sum_{j=2}^{n} z^j - \sum_{j=2}^{n} \vec{x}_t^j - \vec{B}_t^{n,n+1} + \vec{B}_t^{12} + \vec{a}_t^1] \wedge K_n \tag{A5}$$

$$\begin{aligned}
\vec{q}_t^{n,n+1} &= \vec{x}_t^{n+1} \wedge [\vec{a}_t^n + \vec{B}_t^{n,n+1}] && \text{(A6)} \\
&= \vec{x}_t^{n+1} \wedge [[\sum_{j=2}^{n} z^j - \sum_{j=2}^{n} \vec{x}_t^j - \vec{B}_t^{n,n+1} + \vec{B}_t^{12} + \vec{a}_t^1] \wedge K_n + \vec{B}_t^{n,n+1}] && \text{(A7)} \\
&= \vec{x}_t^{n+1} \wedge [[\sum_{j=1}^{n} z^j - \sum_{j=1}^{n} \vec{x}_t^j - \vec{B}_t^{n,n+1}] \wedge K_n + \vec{B}_t^{n,n+1}] && \text{(A8)} \\
&= \vec{x}_t^{n+1} \wedge [[Z^j - X_t^j - \vec{B}_t^{n,n+1}] \wedge K_n + \vec{B}_t^{n,n+1}] && \text{(A9)} \\
&= \vec{x}_t^{n+1} \wedge [[Z^j - X_t^j] \wedge [K_n + \vec{B}_t^{n,n+1}]] && \text{(A10)} \\
&= [Z^j - X_t^j] \wedge K_n \wedge \vec{x}_t^{n+1} = a_t^n && \text{(A11)}
\end{aligned}$$

where equation (A7) comes from equation (A5), equation (A8) comes from $\vec{a}_t^1 = [z^1 - \vec{x}_t^1 - \vec{B}_t^{12}] \wedge K_1 = z^1 - \vec{x}_t^1 - \vec{B}_t^{12}$ for $K_1 = \infty$, and equations (A9)-(A11) come from the definitions in Section 2. ■

Proof of Lemma 13 This results directly from Lemma 3. ■

Proof of Lemma 14 This results directly from Lemma 4. ■

B Payments for Incentive Compatibility Mechanism

The value functions of the incentive compatible mechanism from Section 5 are "accounting corrected" due to the payments between the retailer and supplier. Also, each value function has a dependency on the other firm's policy, so C_n^1 depends on Y^2 and C_n^2 depends on Y^1:

$$C_n^1(X^1, X^2) = \min_{X^1 \le Y^1} \begin{cases} L_1(Y^1) + g_n^{2U}(Y^2|Y^1) + \alpha \mathrm{E} C_{n-1}^1(Y^1 - D, Y^2 - D) & \text{if } X^2 \ge S_n^{1*} \\ L_1(X^2) + g_n^{2U}(Y^2|Y^1) - g_n^{1U}(X^2) + \alpha \mathrm{E} C_{n-1}^1(X^2 - D, Y^2 - D) & \text{if } X^2 < S_n^{1*} \end{cases}$$

$$C_n^2(X^1, X^2) = \min_{X^2 \le Y^2} \begin{cases} L_2(Y^2) + g_n^{1U}(X^2) + \alpha \mathrm{E} C_{n-1}^2(Y^1 - D, Y^2 - D) & \text{if } \max(X^1, S_n^{1*}) + K \ge S_n^{2*} \\ L_2(\max(X^1, S_n^{1*}) + K) - g_n^{2U}(Y^2|Y^1) + g_n^{1U}(X^2) \\ \quad + \alpha \mathrm{E} C_{n-1}^2(Y^1 - D, \max(X^1, S_n^{1*}) + K - D) & \text{if } \max(X^1, S_n^{1*}) + K < S_n^{2*} \end{cases}$$

The following tables contain the payments in the incentive compatibility mechanism described in Section 5. As distinct from Lee and Whang (1999), there are potentially payments from the retailer to the supplier and vice versa.

We consider two situations: the first table shows the payments when the inventory targets intersect within the band while the second shows the payments when they inventory targets intersect outside the band. Each cell contains two functions: the upper function is the amount the retailer pays to the supplier, and the lower function is the amount the supplier pays to the retailer.

Penalty functions for target within the band ($S_n^{2*} - S_n^{1*} \leq K$):

	$X^1 \geq S_n^{1*}, X^1 + K \geq S_n^{2*}$	$X^1 < S_n^{1*}, S_n^{1*} + K \geq S_n^{2*}$
$X^2 \geq S_n^{1*}$	0 0	0 0
$X^2 < S_n^{1*}$ $X^2 + K \geq S_n^{2*}$ $S_n^{1*} + K \geq S_n^{2*}$		0 $f_n^1(X^2) - f_n^1(S_n^{1*})$
$X^2 < S_n^{1*}$ $X^2 + K < S_n^{2*}$ $S_n^{1*} + K \geq S_n^{2*}$		$g_n^{2U}(Y^2\|X^2)$ $f_n^1(X^2) - f_n^1(S_n^{1*}) + g_n^{2U}(X^2 + K) - g_n^{2U}(S_n^{2*})$

Penalty functions for target outside the band ($S_n^{2*} - S_n^{1*} > K$):

	$X^1 \geq S_n^{1*}, X^1 + K \geq S_n^{2*}$	$X^1 \geq S_n^{1*}, X^1 + K < S_n^{2*}$	$X^1 < S_n^{1*}, S_n^{1*} + K < S_n^{2*}$
$X^2 \geq S_n^{1*}$	0 0	$g_n^{2U}(Y^2\|X^1)$ 0	$g_n^{2U}(Y^2\|S_n^{1*})$ 0
$X^2 < S_n^{1*}$ $X^2 + K < S_n^{2*}$ $S_n^{1*} + K < S_n^{2*}$			$g_n^{2U}(Y^2\|X^2)$ $f_n^1(X^2) - f_n^1(S_n^{1*}) + g_n^{2U}(X^2 + K) - g_n^{2U}(S_n^{1*} + K)$