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ONLY AVAILABLE IN ELECTRONIC FORM

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Online Appendix - Not Intended for Print Publication

Literature: Consumer Behavior

The consumer behavior literature contains interesting and relevant work that provides further empirical motivation for our work. Here we briefly survey consumer behavior literature on: frequency of stockouts; estimating shortage costs; consumer behavior upon experiencing stockouts; and advertising models. We compare and contrast the results of this literature with our model.

Since our paper models consumer behavior after a stockout, it is important to know whether stockouts are indeed relevant in practice. Stockout levels can vary between 10-30% in retail settings (as surveyed in Fitzsimons, 2000), between 8.2% (Fitzsimons, 2000) and 18% (Balachander and Farquhar, 1994) in supermarkets, 8-10% in grocery goods, and 20-40% in catalog items (Anderson et al., 2006). Thus, the incorporation of the consumers' activities subsequent to experiencing a stockout are indeed highly relevant.

Establishing a precise estimate for a unit shortage cost can be a challenging exercise, as illustrated by Oral et al. (1972) and Anderson et al. (2006). The primary reason for this is that most retail establishments record actual sales, not the primary demand of consumers. The ultimate sales can, of course, be the culmination of a brand substitution, a size substitution, or some other mediating activity between the demand and the sale, in addition to the lack of a sale altogether which will oftentimes not be registered at all. In their study of mail order catalogs, Anderson et al. (2006) find a firm has an 86% probability of earning revenue if an item is in stock but this falls to 62% if the item is not in stock. They also find there are diminished purchases by customers who experience a stockout, quantified at \$6 per customer for one year and as much as \$23 per customer in the long-term. These values could be interpreted as the difference in values of a "committed" customer and an "latent" customer, a quantity we are able to isolate in our subsequent analysis.

Balachander and Farquhar (1994) find holding less stock can potentially reduce price competition between firms, which can offset the reduced sales from stockouts; we take retail prices as given and fixed. Charlton and Ehrenberg (1976) find no long-term market share or category sales reduction in their experiment with detergents stockouts, whereas Motes and Castleberry (1985) find reduced long-term market shares but restored levels of category sales. The retailer(s) in our models deal(s) with a single item, so substitution between brands at a single location is not possible. The differing levels of "blame" and "forgiveness" found in these empirical studies can be accommodated in our models since they are parameterized.

Straughn (1991) (using scanner data) and Fitzsimons (2000) (using experiments) find there are sustained market share effects of stockouts. Schary and Christopher (1979) report that upon experiencing a stockout 48% of British supermarket shoppers decided to shop elsewhere, 30% chose not to purchase at all or postpone their purchase to a subsequent visit, 17% switched brands, and 5% substituted a different size. Emmelhainz et al. (1991) find 14% shopped at another store, 32% switched brands, and 41% substituted a different size or variety. Looking at apparel sales, Kalyanam et al. (2007) find there is little size substitution and focus primarily on the effect of key item stockouts on ancillary items. The message of this is that there is underlying variation. These quantities are represented by the flow parameters in our models and are relatively unrestricted. Liberopoulos and Tsikis (2005) find similar effects at the wholesale level, showing stockouts negatively affect future demand, reducing the value of future orders and lengthening the time before the next order.

We make no assumption that the availability of inventory on the shelf will have a stimulating effect upon the primary demand, a common assumption in the marketing literature. In addition, we do not distinguish between the inventory holdings in different locations. We assume if an item is in stock, it is available to be sold, although we recognize stock could be "shrunk" (i.e., stolen), in different locations (e.g., retail shelf, retail stockroom, warehouse, in transit), or misplaced. Ton and Raman (2005) approximately quantify misplaced SKUs at 3%. Such considerations could be the subject of future research.

The final elements of the consumer behavior literature which are relevant for us to consider pertain to advertising models. We condense this vast literature into some classical and recent references. The advertising response function is a standard tool, measuring the consumer response for a given advertising expenditure. S-shaped advertising response functions (see Sasieni, 1971, and Feinberg, 2001) are commonly believed to represent the effect of these promotions. Specifically, there is little effect in the market for some initial expenditures (the advertising "threshold"), but then a substantial effect is observed for further outlays, which then tapers off in diminishing returns for higher spending; the overall shape of this curve is S-shaped. There is debate as to the existence or extent of the initial threshold (see Vakratsas et al., 2004), but there tends to be a consensus that there are diminishing returns to scale at higher advertising spending (i.e., a convex increasing advertising cost function). Our assumptions on advertising will indeed incorporate diminishing returns to scale. Nguyen and Shi (2006) incorporate such diminishing returns in a competitive advertising model where market sizes are affected.

Establishment of Basis for Theorem 1

In the basis for induction, we need to show

- $V_T(x,\theta,\beta) = a_T\theta + b_T\beta + c_T$ for $x \le y_f(z_T^*,\theta)$ and is bounded above by $V_T(y_f(z_T^*,\theta),\theta,\beta)$ for $x > y_f(z_T^*,\theta)$;
- $z_{my}^* \le z_T^*$;
- $0 \le a_T - b_T \le a_{T+1} - b_{T+1}$, $a_T \le a_{T+1}$, and $b_T \le b_{T+1}$.

However, the first and second bullets follow by arguments identical to those in the proof of Theorem 1 with substitution of T for t.

Recall from Assumption 5,

$$\begin{aligned} a_{T+1} &= \frac{p_2\tilde{L}(z_{my}^*) + rp_1}{(1-\alpha)^2}, && (47) \\ b_{T+1} &= \alpha a_{T+1}, \text{ and } c_{T+1} = 0. && (48) \end{aligned}$$

So that,

$$a_{T+1} - b_{T+1} = (1-\alpha)a_{T+1}.$$

Further,

$$\begin{aligned} m_T &= \max_{0\le z\le 1}(\tilde{L}(z) - \alpha S(z)(a_{T+1} - b_{T+1})) \\ a_T &= p_2 m_T + \alpha a_{T+1} + rp_1 \\ b_T &= \max_{0\le\rho\le 1}(-C(\rho) + \alpha\rho(a_{T+1} - b_{T+1})) + \alpha b_{T+1} \\ c_T &= \max_{\nu\ge 0}(\alpha a_{T+1}\nu - K(\nu)) + p_3 m_T + \alpha c_{T+1}. \end{aligned}$$

Then,

$$\begin{aligned} a_T &\le p_2\tilde{L}(z_{my}^*) + \alpha a_{T+1} + rp_1 \\ &= \frac{(p_2\tilde{L}(z_{my}^*) + rp_1)((1-\alpha)^2 + \alpha)}{(1-\alpha)^2} \\ &= a_{T+1}(1 - \alpha(1-\alpha)) \\ &\le a_{T+1} \end{aligned}$$

Using the non-negativity of $C(\cdot)$ and then $\rho \le 1$,

$$b_T \le \alpha a_{T+1} = b_{T+1}.$$

Substituting $\rho = 0$ as a lower bound,

$$b_T \ge \alpha b_{T+1}$$

Further,

$$a_T - b_T \le p_2\tilde{L}(z_{my}^*) + rp_1 + \alpha(a_{T+1} - b_{T+1})$$

$$
\begin{aligned}
&= (1-\alpha)^2 a_{T+1} + \alpha(1-\alpha)a_{T+1} \\
&= (1-\alpha)a_{T+1} \\
&= a_{T+1} - b_{T+1}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
a_T - b_T &\geq p_2\tilde{L}(z^*_{my}) + rp_1 - \alpha p_2 S(z^*_{my})(a_{T+1} - b_{T+1}) + \alpha a_{T+1} \\
&\quad -(\alpha\rho^*_T(a_{T+1} - b_{T+1}) + \alpha b_{T+1}) \\
&= p_2\tilde{L}(z^*_{my}) + rp_1 + \alpha(a_{T+1} - b_{T+1})(1 - \rho^*_T - p_2 S(z^*_{my})) \\
&\geq 0.
\end{aligned}
$$

Where the final inequality follows by Assumption 6. This has established the basis. Q.E.D.

Extension to the Proof of Theorem 1 When There is No Assumption on Initial Inventory

Recall that $y^{-1}(x,\theta) = \Phi((x - p_1\theta)/(p_2\theta + p_3))$. We define a function $h(x,\theta)$ to be *nonincreasing in* $y^{-1}(x,\theta)$ if for every (x^1,θ^1) and (x^2,θ^2) such that $y^{-1}(x^1,\theta^1) \leq y^{-1}(x^2,\theta^2)$ we have that $h(x^1,\theta^1) \geq h(x^2,\theta^2)$. In lieu of the first inductive assumption for period t, $1 \leq t \leq T-1$, assume:

- $V_{t+1}(x,\theta,\beta) = a_{t+1}\theta + b_{t+1}\beta + c_{t+1} + h_{t+1}(x,\theta)$, where $h_{t+1}(x,\theta) = 0$ for $x \leq y_f(z^*_{t+1},\theta)$ and is non-negative and nonincreasing in $y^{-1}(x,\theta)$ (and is independent of β) for $x > y_f(z^*_{t+1},\theta)$.

This is a slightly stronger condition that implies the original condition. Is is true for the basis (by assumption) with $h_{T+1}(x,\theta) \equiv 0$.

Consider the case where $z^*_t < y^{-1}(x_t,\theta_t)$ (i.e., ordering up to z^*_t is not feasible). Then, analogous to in the proof of Theorem 1,

$$
\begin{aligned}
V_t(x,\theta,\beta) = {} & (p_2\theta + p_3) \max_{z \geq y^{-1}(x,\theta)} (f_t(z) + \mathrm{E}[h_{t+1}(x_{t+1},\theta_{t+1})]) + \theta(rp_1 + \alpha a_{t+1}) + \\
& \beta \max_{0 \leq \rho \leq 1} (-C(\rho) + \alpha(\rho_t(a_{t+1} - b_{t+1}) + b_{t+1})) + \alpha c_{t+1} + \max_{\nu \geq 0}(\alpha a_{t+1}\nu - K(\nu)).
\end{aligned}
$$

If suffices to show that $\mathrm{E}[h_{t+1}(x_{t+1},\theta_{t+1})]$ is nonincreasing in the decision z. Then the concave nature of $f_t(z)$ implies that $f_t(z) + \mathrm{E}[h_{t+1}(x_{t+1},\theta_{t+1})]$ is minimized at $y^{-1}(x,\theta)$ (and hence ordering nothing when above the desired base-stock is optimal). Further, define

$$
h_t(x,\theta) = f_t(y^{-1}(x,\theta)) - f_t(z^*_t) + \mathrm{E}[h_{t+1}(x_{t+1},\theta_{t+1})],
$$

where (x_{t+1},θ_{t+1}) are determined by ordering up to $y^{-1}(x,\theta)$. Then $V_t(x,\theta,\beta) = a_t\theta + b_t\beta + c_t + h_t(x,\theta)$. By combining the above with the previous analysis in the proof of Theorem 1 in the appendix, $h_t(x,\theta) = 0$ for $x \leq y_f(z^*_t,\theta)$. Further, it is clearly non-negative, independent of β_t, and nonincreasing in $y^{-1}(x,\theta)$ (if we have shown that $\mathrm{E}[h_{t+1}(x_{t+1},\theta_{t+1})]$ is nonincreasing in $z = y^{-1}(x,\theta)$) for $x > y_f(z_t^*,\theta)$.

Thus it remains to show that $\mathrm{E}[h_{t+1}(x_{t+1},\theta_{t+1})]$ is indeed nonincreasing in the decision z_t. Fix the demand realization ε_t. If we can show for every realization of ε_t that $y^{-1}(x_{t+1},\theta_{t+1})$ is nondecreasing in z_t then, since $h_{t+1}(x,\theta)$ is nonincreasing in $y^{-1}(x,\theta)$ (by the inductive assumption), we have that $\mathrm{E}[h_{t+1}(x_{t+1},\theta_{t+1})]$ is nonincreasing in z_t. Recall that:

$$\begin{aligned} x_{t+1} &= (p_2\theta_t + p_3)(\Phi^{-1}(z_t) - \varepsilon_t + \Gamma_t(\varepsilon_t - \Phi^{-1}(z_t))^+) \\ \theta_{t+1} &= \theta_t - (p_2\theta_t + p_3)\Lambda_t\Gamma_t(\varepsilon_t - \Phi^{-1}(z_t))^+ + R_t(\rho_t)\beta_t + U_t(\nu_t). \end{aligned}$$

Then

$$\begin{aligned} & y^{-1}(x_{t+1},\theta_{t+1}) \\ &= \Phi\left(\frac{x_{t+1} - p_1\theta_{t+1}}{p_2\theta_{t+1} + p_3}\right) \\ &= \Phi\left(\frac{(p_2\theta_t + p_3)(\Phi^{-1}(z_t) - \varepsilon_t + \Gamma_t(\varepsilon_t - \Phi^{-1}(z_t))^+(1 - p_1\Lambda_t)) - p_1(\theta_t + R_t(\rho_t)\beta_t + U_t(\nu_t))}{p_2(\theta_t - (p_2\theta_t + p_3)\Lambda_t\Gamma_t(\varepsilon_t - \Phi^{-1}(z_t))^+ + R_t(\rho_t)\beta_t + U_t(\nu_t)) + p_3}\right). \end{aligned}$$

If $(\varepsilon_t - \Phi^{-1}(z_t))^+ = 0$ then this is clearly increasing in z_t (locally). Now suppose $(\varepsilon_t - \Phi^{-1}(z_t))^+ > 0$, then

$$y^{-1}(x_{t+1},\theta_{t+1}) = \Phi\left(\frac{(p_2\theta_t + p_3)(\Phi^{-1}(z_t) - \varepsilon_t)(1 - \Gamma_t(1 - p_1\Lambda_t)) - p_1(\theta_t + R_t(\rho_t)\beta_t + U_t(\nu_t))}{(p_2\theta_t + p_3)(\Phi^{-1}(z_t) - \varepsilon_t)p_2\Lambda_t\Gamma_t + p_2(\theta_t + R_t(\rho_t)\beta_t + U_t(\nu_t)) + p_3}\right).$$

Dividing through by $(\Phi^{-1}(z_t) - \varepsilon_t)$ we can see that the numerator is increasing in z_t and the denominator decreasing in z_t making the whole increasing in z_t. As the function is continuous at $\varepsilon_t = \Phi^{-1}(z_t)$, the proof is complete. Q.E.D.

Proof of Lemma 1

By the definition of $\rho_f(\Delta)$,

$$C(\rho_f(\Delta)) - \alpha\Delta\rho_f(\Delta) \le C(\rho) - \alpha\Delta\rho|_{\rho=0} = 0.$$

Using $S(\cdot) \ge 0$ and the definition of z_{my}^* as the maximizer of $\tilde{L}(\cdot)$,

$$T(\Delta) \le p_2\tilde{L}(z_{my}^*) + rp_1 + \alpha\Delta_{MAX},$$

where Δ_{MAX} is some upper bound on Δ. Thus, if a fixed point exists, $\Delta^* = T(\Delta^*)$ and

$$\Delta^* \le p_2\tilde{L}(z_{my}^*) + rp_1 + \alpha\Delta_{MAX}.$$

Thus we can let

$$\Delta_{MAX} = \frac{p_2\tilde{L}(z_{my}^*) + rp_1}{(1-\alpha)} \triangleq \Delta_{max},$$

which yields, for any fixed point Δ^*, $\Delta^* \leq \Delta_{max}$ and we can restrict attention to $\Delta \leq \Delta_{max}$. Now, for $\Delta \geq 0$,

$$\begin{aligned} z_f(\Delta) &= 1 - \frac{h}{\tilde{r} + \alpha\Delta\gamma\lambda + h} \\ &\leq 1 - \frac{h}{\tilde{r} + \alpha\Delta_{max}\gamma\lambda + h} = z_{max}. \end{aligned}$$

Further, for $0 < \Delta \leq \Delta_{max}$,

$$\begin{aligned} \rho_f(\Delta) &= C'^{-1}(\alpha\Delta) \\ &\leq C'^{-1}(\alpha\Delta_{max}) = \rho_{max}. \end{aligned}$$

If $\Delta \leq 0$ (which will actually be shown to be excluded) then $z_f(\Delta) \leq z^*_{my} \leq z_{max}$ and $\rho_f(\Delta) = 0 \leq \rho_{max}$.

For $\Delta \geq 0$,

$$\frac{dz_f(\Delta)}{d\Delta} = \frac{h\alpha\lambda\gamma}{(\tilde{r} + \alpha\lambda\gamma\Delta + h)^2} > 0$$

and for $0 < \rho_f(\Delta) < 1$, by the inverse function theorem,

$$\frac{d\rho_f(\Delta)}{d\Delta} = \frac{\alpha}{C''(\rho_f(\Delta))} > 0.$$

Thus, both $z_f(\Delta)$ and $\rho_f(\Delta)$ are nondecreasing in Δ and hence $z_f(\Delta) \geq z^*_{my}$ for $\Delta \geq 0$. Further,

$$\begin{aligned} \frac{dT(\Delta)}{d\Delta} &= p_2 \frac{d}{dz}\left[\tilde{L}(z) - \alpha\Delta S(z)\right]\Big|_{z=z_f(\Delta)} \frac{dz_f(\Delta)}{d\Delta} + \frac{d}{d\rho}\left[C(\rho) - \alpha\Delta\rho\right]\Big|_{\rho=\rho_f(\Delta)} \frac{d\rho_f(\Delta)}{d\Delta} \\ &\quad + \alpha(1 - p_2 S(z_f(\Delta)) - \rho_f(\Delta)) \\ &= \alpha(1 - p_2 S(z_f(\Delta)) - \rho_f(\Delta)) \geq \alpha(1 - p_2 S(z^*_{my}) - \rho_{max}) \geq 0 \end{aligned}$$

where the equality follows since $\frac{d}{dz}\left[\tilde{L}(z) - \alpha\Delta S(z)\right]\Big|_{z=z_f(\Delta)} = 0$ and, by the strict convexity of $C(\cdot)$, either $\frac{d}{d\rho}\left[C(\rho) - \alpha\Delta\rho\right]\Big|_{\rho=\rho_f(\Delta)} = 0$ or $\frac{d\rho_f(\Delta)}{d\Delta} = 0$. Thus $0 \leq \frac{dT(\Delta)}{d\Delta} < 1$ and hence $T(\cdot)$ is a contraction mapping with a unique fixed point Δ^*. Further, $T(0) = p_2\tilde{L}(z^*_{my}) + rp_1 > 0$ and $\frac{dT(\Delta)}{d\Delta} \geq 0$ implies that $\Delta^* > 0$. Q.E.D.

Proof of Proposition 1

Define $G(\Delta^*) = \Delta^* - T(\Delta^*)$. In preparation for applying the implicit function theorem, let us differentiate G:

$$\begin{aligned} \frac{\partial}{\partial\Delta^*}G(\Delta^*) &= 1 - p_2 \frac{\partial}{\partial z}\left[\tilde{L}(z) - \alpha\Delta^* S(z)\right]\Big|_{z=z_f(\Delta^*)} \frac{\partial z_f(\Delta^*)}{\partial\Delta^*} - \frac{\partial}{\partial\rho}\left[C(\rho) - \alpha\Delta^*\rho\right]\Big|_{\rho=\rho_f(\Delta^*)} \frac{\partial\rho_f(\Delta^*)}{\partial\Delta^*} \\ &\quad - \alpha(1 - p_2 S(z_f(\Delta^*)) - \rho_f(\Delta^*)) \\ &= 1 - \alpha(1 - p_2 S(z_f(\Delta^*)) - \rho_f(\Delta^*)) \geq 0 \end{aligned}$$

$$\begin{aligned}
-\frac{\partial}{\partial r}G(\Delta^*) &= \frac{\partial}{\partial r}T(\Delta^*) \\
&= p_2\,\frac{\partial}{\partial z}\left[\tilde{L}(z)-\alpha\Delta^* S(z)\right]\Bigg|_{z=z_f(\Delta^*)}\frac{\partial z_f(\Delta^*)}{\partial r}+p_1+p_2\frac{\partial}{\partial r}\tilde{L}(z) \\
&= p_1-p_2(1-\alpha(1-\gamma))\mathrm{E}[(\varepsilon-\Phi^{-1}(z_f(\Delta^*)))^+]\geq 0 \qquad \text{if } p_1\geq p_2 \\
-\frac{\partial}{\partial h}G(\Delta^*) &= \frac{\partial}{\partial h}T(\Delta^*) \\
&= p_2\frac{\partial}{\partial z}\left[\tilde{L}(z)-\alpha\Delta^* S(z)\right]\Bigg|_{z=z_f(\Delta^*)}\frac{\partial z_f(\Delta^*)}{\partial h}+p_2\frac{\partial}{\partial h}\tilde{L}(z) \\
&= -p_2\mathrm{E}[(\Phi^{-1}(z_f(\Delta^*))-\varepsilon)^+]\leq 0 \\
-\frac{\partial}{\partial \gamma}G(\Delta^*) &= \frac{\partial}{\partial \gamma}T(\Delta^*) \\
&= p_2\frac{\partial}{\partial z}\left[\tilde{L}(z)-\alpha\Delta^* S(z)\right]\Bigg|_{z=z_f(\Delta^*)}\frac{\partial z_f(\Delta^*)}{\partial \gamma}+p_2\frac{\partial}{\partial \gamma}\tilde{L}(z)-\alpha p_2\Delta^*\lambda\mathrm{E}[(\varepsilon-\Phi^{-1}(z_f(\Delta^*)))^+] \\
&= -\alpha p_2(\Delta^*\lambda+r)\mathrm{E}[(\varepsilon-\Phi^{-1}(z_f(\Delta^*)))^+]\leq 0 \\
-\frac{\partial}{\partial \lambda}G(\Delta^*) &= \frac{\partial}{\partial \lambda}T(\Delta^*) \\
&= p_2\frac{\partial}{\partial z}\left[\tilde{L}(z)-\alpha\Delta^* S(z)\right]\Bigg|_{z=z_f(\Delta^*)}\frac{\partial z_f(\Delta^*)}{\partial \lambda}-\alpha p_2\Delta^*\gamma\mathrm{E}[(\varepsilon-\Phi^{-1}(z_f(\Delta^*)))^+] \\
&= -\alpha p_2\Delta^*\gamma\mathrm{E}[(\varepsilon-\Phi^{-1}(z_f(\Delta^*)))^+]\leq 0 \\
-\frac{\partial}{\partial \alpha}G(\Delta^*) &= \frac{\partial}{\partial \alpha}T(\Delta^*) \\
&= p_2\frac{\partial}{\partial z}\left[\tilde{L}(z)-\alpha\Delta^* S(z)\right]\Bigg|_{z=z_f(\Delta^*)}\frac{\partial z_f(\Delta^*)}{\partial \alpha}+\frac{\partial}{\partial \rho}\left[C(\rho)-\alpha\Delta^*\rho\right]\Big|_{\rho=\rho_f(\Delta^*)}\frac{\partial \rho_f(\Delta^*)}{\partial \alpha} \\
&\quad +p_2\frac{\partial}{\partial \alpha}\tilde{L}(z)+\Delta^*(1-p_2S(z_f(\Delta^*))-\rho_f(\Delta^*)) \\
&= rp_2(1-\gamma)\mathrm{E}[(\varepsilon-\Phi^{-1}(z_f(\Delta^*)))^+]+\Delta^*(1-p_2S(z_f(\Delta^*))-\rho_f(\Delta^*)) \\
&\geq rp_2(1-\gamma)\mathrm{E}[(\varepsilon-\Phi^{-1}(z_f(\Delta^*)))^+]+\Delta^*(1-p_2S(z_{my})-\rho_{max})\geq 0
\end{aligned}$$

Applying the implicit function theorem:

$$\frac{\partial\Delta^*}{\partial r}=\frac{-\frac{\partial G}{\partial r}}{\frac{\partial G}{\partial\Delta^*}}\geq 0,\quad \frac{\partial\Delta^*}{\partial h}=\frac{-\frac{\partial G}{\partial h}}{\frac{\partial G}{\partial\Delta^*}}\leq 0,\quad \frac{\partial\Delta^*}{\partial \gamma}=\frac{-\frac{\partial G}{\partial \gamma}}{\frac{\partial G}{\partial\Delta^*}}\leq 0,\quad \frac{\partial\Delta^*}{\partial \lambda}=\frac{-\frac{\partial G}{\partial \lambda}}{\frac{\partial G}{\partial\Delta^*}}\leq 0,\quad \frac{\partial\Delta^*}{\partial \alpha}=\frac{-\frac{\partial G}{\partial \alpha}}{\frac{\partial G}{\partial\Delta^*}}\geq 0.$$

The optimal solution of the stocking level is:

$$z_f(\Delta^*)=1-\frac{h}{\tilde{r}+\alpha\lambda\gamma\Delta^*+h}$$

Define $g(\Delta^*)=\tilde{r}+\alpha\Delta^*\lambda\gamma+h$

$$\begin{aligned}
\frac{\partial z_f(\Delta^*)}{\partial\Delta^*} &= \frac{h\alpha\lambda\gamma}{(g(\Delta^*))^2}\geq 0 \\
\frac{\partial z_f(\Delta^*)}{\partial r} &= \frac{h(1-\alpha(1-\gamma))}{(g(\Delta^*))^2}+\frac{h\alpha\lambda\gamma}{(g(\Delta^*))^2}\frac{\partial\Delta^*}{\partial r}\geq 0 \\
\frac{\partial z_f(\Delta^*)}{\partial h} &= \frac{-\tilde{r}-\alpha\Delta^*\lambda\gamma-h+h}{(g(\Delta^*))^2}+\frac{h\alpha\lambda\gamma}{(g(\Delta^*))^2}\frac{\partial\Delta^*}{\partial h}\leq 0
\end{aligned}$$

$$\frac{\partial z_f(\Delta^*)}{\partial \alpha} = \frac{-hr(1-\gamma)+h\lambda\gamma\Delta^*}{(g(\Delta^*))^2} + \frac{h\alpha\lambda\gamma}{(g(\Delta^*))^2}\frac{\partial \Delta^*}{\partial \alpha} \geq 0,$$

where the final inequality follows from the sufficient condition in the theorem statement. The optimal solution to the incentive decision is:

$$\rho_f(\Delta^*) = \min(C'^{-1}(\alpha\Delta^*), 1).$$

Clearly, $\rho_f(\Delta^*)$ may adopt the value of 1 or a lesser (positive) value. Taking the derivative of 1 with respect to various parameters will yield 0, so we will hereafter assume $\rho_f(\Delta^*) = C'^{-1}(\alpha\Delta^*)$ for the remainder of this analysis.

$$\begin{aligned}
\frac{\partial \rho_f(\Delta^*)}{\partial \Delta^*} &= \frac{\alpha}{C''(\rho_f(\Delta^*))} \geq 0 \\
\frac{\partial \rho_f(\Delta^*)}{\partial r} &= \frac{\alpha}{C''(\rho_f(\Delta^*))}\frac{\partial \Delta^*}{\partial r} \geq 0 \\
\frac{\partial \rho_f(\Delta^*)}{\partial h} &= \frac{\alpha}{C''(\rho_f(\Delta^*))}\frac{\partial \Delta^*}{\partial h} \leq 0 \\
\frac{\partial \rho_f(\Delta^*)}{\partial \gamma} &= \frac{\alpha}{C''(\rho_f(\Delta^*))}\frac{\partial \Delta^*}{\partial \gamma} \leq 0 \\
\frac{\partial \rho_f(\Delta^*)}{\partial \lambda} &= \frac{\alpha}{C''(\rho_f(\Delta^*))}\frac{\partial \Delta^*}{\partial \lambda} \leq 0 \\
\frac{\partial \rho_f(\Delta^*)}{\partial \alpha} &= \frac{\Delta^*}{C''(\rho_f(\Delta^*))} + \frac{\alpha}{C''(\rho_f(\Delta^*))}\frac{\partial \Delta^*}{\partial \alpha} \geq 0
\end{aligned}$$

Q.E.D.

Proof of Theorem 3

In the basis for induction, we will show $(i \neq j)$

- $V_T^i(x^i, x^j, \theta^i, \theta^j) = a_T^i\theta^i + b_T^i\theta^j + c_T$ for $x^i \leq y_f^i(z_T^{i*}, \theta^i)$ and is bounded above by $V_T^i(y_f^i(z_T^{i*}, \theta^i), x^j, \theta^i, \theta^j)$;
- $z_{my}^{i*} \leq z_T^{i*}$;
- $0 \leq a_T^i - b_T^i \leq a_{T+1}^i - b_{T+1}^i$; and
- $a_T^i \leq a_{T+1}^i$ and $b_T^i \leq b_{T+1}^i$.

Now,

$$\begin{aligned}
& a_{T+1}^i \mathrm{E}\theta_{T+1}^i + b_{T+1}^i \mathrm{E}\theta_{T+1}^j \\
&= a_{T+1}^i(\theta_T^i - (p_2^i\theta_T^i + p_3^i)S^i(z^i) + (p_2^j\theta_T^j + p_3^j)S^j(z^j)) \\
&\quad + b_{T+1}^i(\theta_T^j - (p_2^j\theta_T^j + p_3^j)S^j(z^j) + (p_2^i\theta_T^i + p_3^i)S^i(z^i)) \\
&= a_{T+1}^i\theta_T^i - (p_2^i\theta_T^i + p_3^i)S^i(z^i)(a_{T+1}^i - b_{T+1}^i) + b_{T+1}^i\theta_T^j + (p_2^j\theta_T^j + p_3^j)S^j(z^j)(a_{T+1}^i - b_{T+1}^i)
\end{aligned}$$

For period T,

$$\begin{aligned}
V_T^i(x^i,x^j,\theta^i,\theta^j) &= \max_{z^i \geq y^{-1}(x^i,\theta^i)} \left[(p_2^i\theta^i + p_3^i)\tilde{L}^i(z^i) + rp_1^i\theta^i + \alpha(a_{T+1}^i \mathrm{E}\theta_{T+1}^i + b_{T+1}^i \mathrm{E}\theta_{T+1}^j + c_{T+1}^i) \right] \\
&= \max_{z^i \geq y^{-1}(x^i,\theta^i)} \begin{bmatrix} (p_2^i\theta^i + p_3^i)(\tilde{L}^i(z^i) - \alpha S^i(z^i)(a_{T+1}^i - b_{T+1}^i)) + (a_{T+1}^i + rp_1)\theta^i \\ +\alpha(p_2^j\theta_T^j + p_3^j)S^j(z^j)(a_{T+1}^i - b_{T+1}^i) + \alpha b_{T+1}^i\theta_T^j + \alpha c_{T+1}^i \end{bmatrix}
\end{aligned}$$

Thus $V_T^i(x^i,x^j,\theta^i,\theta^j) = a_T^i\theta^i + b_T^i\theta^j + c_T^i$ for $x^i \leq y_f^i(z_T^{i*},\theta^i)$. For $x^i > y_f^i(z_T^{i*},\theta^i)$, $V_T^i(x^i,x^j,\theta^i,\theta^j) \leq V_T^i(y_f^i(z_T^{i*},\theta^i),x^j,\theta^i,\theta^j) = a_T^i\theta^i + b_T^i\theta^j + c_T^i$ by the optimality of z_T^{i*} in the above.

We have immediately from assumption 10,

$$a_{T+1}^i - b_{T+1}^i = \frac{(p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i)(1-\alpha)}{(1-\alpha)^2} \geq 0.$$

Thus, from its definition, $z_T^{i*} \geq z_{my}^{i*}$. Further,

$$\begin{aligned}
a_T^i &\leq p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i + \alpha a_{T+1}^i \\
&= \frac{(p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i)((1-\alpha)^2 + \alpha)}{(1-\alpha)^2} \\
&= a_{T+1}^i(1-\alpha+\alpha^2) \\
&\leq a_{T+1}^i
\end{aligned}$$

From the recursive definitions,

$$\begin{aligned}
b_T^i &\leq \alpha(a_{T+1}^i - b_{T+1}^i) + \alpha b_{T+1}^i = \alpha a_{T+1}^i = b_{T+1}^i \\
b_T^i &\geq \alpha b_{T+1}^i
\end{aligned}$$

$$\begin{aligned}
a_T^i - b_T^i &\leq p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i + \alpha(a_{T+1}^i - b_{T+1}^i) \\
&= p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i + \alpha(1-\alpha)a_{T+1}^i \\
&= (p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i)\left(1 + \frac{\alpha(1-\alpha)}{(1-\alpha)^2}\right) \\
&= \frac{p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i}{1-\alpha} \\
&= a_{T+1}^i - b_{T+1}^i.
\end{aligned}$$

$$\begin{aligned}
a_T^i - b_T^i &\geq p_2^i\tilde{L}^i(z_T^{i*}) + r^ip_1^i - \alpha(a_{T+1}^i - b_{T+1}^i)S^i(z_T^{i*}) + \alpha a_{T+1}^i \\
&\quad -(\alpha(a_{T+1}^i - b_{T+1}^i)p_2^jS^j(z_T^{j*}) + \alpha b_{T+1}^i) \\
&= p_2^i\tilde{L}^i(z_{my}^{i*}) + r^ip_1^i + \alpha(a_{T+1}^i - b_{T+1}^i)(1 - p_2^iS^i(z_{my}^{i*}) - p_2^jS^j(z_{my}^{j*})) \\
&\geq 0.
\end{aligned}$$

from assumption 10. This has established the basis.

For period t, assume $(i \neq j)$

- $V^i_{t+1}(x^i, x^j, \theta^i, \theta^j) = a^i_{t+1}\theta^i + b^i_{t+1}\theta^j + c_{t+1}$ for $x^i \leq y^i_f(z^{i*}_{t+1}, \theta^i)$ and is bounded above by $V^i_{t+1}(y^i_f(z^{i*}_{t+1}, \theta^i), x^j, \theta^i, \theta^j)$ for $x^i > y^i_f(z^{i*}_{t+1}, \theta^i)$;
- $z^{i*}_{my} \leq z^{i*}_{t+1}$;
- $0 \leq a^i_{t+1} - b^i_{t+1} \leq a^i_{t+2} - b^i_{t+2}$; and
- $a^i_{t+1} \leq a^i_{t+2}$ and $b^i_{t+1} \leq b^i_{t+2}$.

Along any sample path

$$V^i_{t+1}(x^i_{t+1}, x^j_{t+1}, \theta^i_{t+1}, \theta^j_{t+1}) \begin{cases} = a^i_{t+1}\theta^i_{t+1} + b^i_{t+1}\theta^j_{t+1} + c^i_{t+1} & x^i_{t+1} \leq y^i_f(z^{i*}_{t+1}, \theta^i_{t+1}) \\ \leq a^i_{t+1}\theta^i_{t+1} + b^i_{t+1}\theta^j_{t+1} + c^i_{t+1} & x^i_{t+1} > y^i_f(z^{i*}_{t+1}, \theta^i_{t+1}) \end{cases}$$

For $z^i_t \leq z^{i*}_{t+1}$, $V^i_{t+1}(x^i_{t+1}, x^j_{t+1}, \theta^i_{t+1}, \theta^j_{t+1}) = a^i_{t+1}\theta^i_{t+1} + b^i_{t+1}\theta^j_{t+1} + c^i_{t+1}$ if $x^i_{t+1} \leq y^i_f(z^{i*}_{t+1}, \theta^i_{t+1})$. Which is true due to the following reasoning. We have that,

$$x^i_{t+1} = (p^i_2\theta^i_t + p^i_3)(\Phi^{-1}_i(z^i_t) - \varepsilon^i_t + \Gamma^i_t(\varepsilon^i_t - \Phi^{-1}_i(z^i_t))^+)$$

and

$$y^i_f(z^{i*}_{t+1}, \theta^i_{t+1}) = p^i_1\theta^i_{t+1} + \Phi^{-1}_i(z^{i*}_{t+1})(p^i_2\theta^i_{t+1} + p^i_3).$$

If $\varepsilon^i_t \leq \Phi^{-1}_i(z^i_t)$, so there are no unsatisfied customers, then $\theta^i_{t+1} \geq \theta^i_t$ and, using the fact that demand must be non-negative, the future desired order-up-to point, $y^i_f(z^{i*}_{t+1}, \theta^i_{t+1})$, is bounded below by

$$p^i_1\theta^i_t + \Phi^{-1}_i(z^i_t)(p^i_2\theta^i_t + p^i_3) \geq (p^i_2\theta^i_t + p^i_3)(\Phi^{-1}_i(z^i_t) - \varepsilon^i_t),$$

where the right hand-side equals future inventory, x^i_{t+1}, (in this case). If $\varepsilon^i_t > \Phi^{-1}(z^i_t)$ then future inventory, x^i_{t+1}, is negative but, again using non-negativity of demand, the future desired order-up-to point, $y^i_f(z^{i*}_{t+1}, \theta^i_{t+1})$, is non-negative. Thus, in both cases, $x^i_{t+1} \leq y^i_f(z^{i*}_{t+1}, \theta^i_{t+1})$.

For $z^i_t > z^{i*}_{t+1}$,

$$\begin{aligned} &\mathrm{E}V^i_{t+1}(x^i_{t+1}, x^j_{t+1}, \theta^i_{t+1}, \theta^j_{t+1}) \\ &\leq a^i_{t+1}\mathrm{E}\theta^i_{t+1} + b^i_{t+1}\mathrm{E}\theta^j_{t+1} + c^i_{t+1} \\ &= a^i_{t+1}\theta^i_t - (p^i_2\theta^i_t + p^i_3)S^i(z^i_t)(a^i_{t+1} - b^i_{t+1}) + b^i_{t+1}\theta^j_t + (p^j_2\theta^j_t + p^j_3)S^j(z^j_t)(a^i_{t+1} - b^i_{t+1}) + c^i_{t+1}. \end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned} &\tilde{L}^i(z^i_t) + \alpha\mathrm{E}V^i_{t+1}(x^i_{t+1}, x^j_{t+1}, \theta^i_{t+1}, \theta^j_{t+1}) \\ &\leq (p^i_2\theta^i_t + p^i_3)f^i_t(z^i_t) + (\alpha a^i_{t+1} + r^i p^i_1)\theta^i_t + \alpha(p^j_2\theta^j_t + p^j_3)S^j(z^j_t)(a^i_{t+1} - b^i_{t+1}) + \alpha b^i_{t+1}\theta^j_t + \alpha c^i_{t+1}, \end{aligned}$$

where

$$f^i_t(z) = \tilde{L}^i(z) - \alpha S^i(z)(a^i_{t+1} - b^i_{t+1}).$$

By the induction assumption, $\arg\max_z f_t^i(z) \le \arg\max_z f_{t+1}^i(z) = z_{t+1}^{i*}$, where the inequality follows from the concavity of $-S^i(z)$ and $\tilde{L}^i(z)$ and the non-decreasing nature of $a_t^i - b_t^i$. Therefore, by the concavity of $f_t^i(z)$, $f_t^i(z_t^i) \le f_t^i(z_{t+1}^{i*})$. Consequently, we can exclude consideration of $z_t^i > z_{t+1}^{i*}$.

Therefore, applying the same logic as in (49),

$$V_t^i(x^i,x^j,\theta^i,\theta^j) = (p_2^i\theta^i + p_3^i)\max_{z\ge y^{-1}(x^i,\theta^i)} f_t^i(z) + (\alpha a_{t+1}^i + r^i p_1^i)\theta_t^i + \alpha(p_2^j\theta_t^j + p_3^j)S^j(z_t^j)(a_{t+1}^i - b_{t+1}^i) + \alpha b_{t+1}^i\theta_t^j + \alpha c_{t+1}^i$$

Now,

$$z_t^{i*} = \arg\max_{0\le z^i\le 1}\left\{\tilde{L}^i(z^i) - \alpha(a_{t+1}^i - b_{t+1}^i)S^i(z^i)\right\} \tag{50}$$
$$a_t^i = p_2^i\tilde{L}^i(z^{i*}) - \alpha p_2^i(a_{t+1}^i - b_{t+1}^i)S^i(z^{i*}) + r^i p_1^i + \alpha a_{t+1}^i \tag{51}$$
$$b_t^i = \alpha p_2^j(a_{t+1}^i - b_{t+1}^i)S^j(z^{j*}) + \alpha b_{t+1}^i \tag{52}$$
$$c_t^i = p_3^i\tilde{L}^i(z^{i*}) - \alpha p_3^i(a_{t+1}^i - b_{t+1}^i)S^i(z^{i*}) + \alpha p_3^j(a_{t+1}^i - b_{t+1}^i)S^j(z^{j*}) + \alpha c_{t+1}^i \tag{53}$$

$$a_t^i = p_2^i(\max_z(\tilde{L}^i(z) - \alpha S^i(z)(a_{t+1}^i - b_{t+1}^i))) + \alpha a_{t+1}^i + r^i p_1^i \tag{54}$$
$$= p_2^i(\tilde{L}^i(z_t^{i*}) - \alpha S^i(z_t^{i*})(a_{t+1}^i - b_{t+1}^i)) + \alpha a_{t+1}^i + r^i p_1^i \tag{55}$$

which is increasing in t since $(1 - S^i(z)) \ge 0$ for all z and $(a_{t+1}^i - b_{t+1}^i)$ is also increasing in t.

$$b_t^i = \alpha p_2^j(a_{t+1}^i - b_{t+1}^i)S^j(z^{j*}) + \alpha b_{t+1}^i \tag{56}$$

which is also increasing along similar reasoning to a_t^i. Further,

$$\begin{aligned}
a_t^i - b_t^i &= p_2^i\tilde{L}^i(z_t^{i*}) + \alpha(a_{t+1}^i - b_{t+1}^i)(1 - p_2^i S^i(z_t^{i*}) - p_2^j S^j(z_t^{j*}))\\
&\le p_2^i\tilde{L}^i(z_t^{i*}) + \alpha(a_{t+2}^i - b_{t+2}^i)(1 - p_2^i S^i(z_t^{i*}) - p_2^j S^j(z_t^{j*}))\\
&\le p_2^i\tilde{L}^i(z_{t+1}^{i*}) + \alpha(a_{t+2}^i - b_{t+2}^i)(1 - p_2^i S^i(z_{t+1}^{i*}) - p_2^j S^j(z_{t+1}^{j*}))\\
&= a_{t+1}^i - b_{t+1}^i
\end{aligned}$$

where the first inequality arises via the induction assumption since $1 - p_2^i S^i(z_t^{i*}) - p_2^j S^j(z_t^{j*}) \ge 1 - p_2^i S^i(z_{my}^{i*}) - p_2^j S^j(z_{my}^{j*}) \ge 0$, and the second inequality arises by the definition of z_{t+1}^{i*}. Finally, $a_t^i - b_t^i \ge 0$ since $1 - p_2^i S^i(z_t^{i*}) - p_2^j S^j(z_t^{j*}) \ge 0$ as above. Q.E.D.

Proof of Lemma 2

Begin by observing:

$$\tilde{S}^i(z) = \gamma^i \mathrm{E}[(\varepsilon^i - \Phi_i^{-1}(z))^+]. \tag{57}$$
$$\partial\tilde{S}^i(z)/\partial z = -\gamma^i \Pr(\varepsilon^i > \Phi_i^{-1}(z))/\phi_i(z) \tag{58}$$
$$g^i(\Delta^i,\lambda^{ij}) = \tilde{r}^i + \alpha\Delta^i\lambda^{ij}\gamma^i + h^i \tag{59}$$

$$\partial g^i(\Delta^i,\lambda^{ij})/\partial\lambda^{ij} = \alpha\Delta^i\gamma^i \tag{60}$$
$$\partial g^i(\Delta^i,\lambda^{ij})/\partial\Delta^i = \alpha\lambda^{ij}\gamma^i \tag{61}$$

Define
$$\tilde z_f^i(\Delta^i,\lambda^{ij}) = 1-\frac{h^i}{g^i(\Delta^i,\lambda^{ij})}.$$
So that, for fixed Δ^i, $\tilde z_f^i(\Delta^i,\lambda^{ij})$ is the z-response function. Then,
$$\frac{\partial\tilde z_f^i(\Delta^i,\lambda^{ij})}{\partial\Delta^i} = \frac{h^i\partial g^i(\Delta^i,\lambda^{ij})/\partial\Delta^i}{(g^i(\Delta^i,\lambda^{ij}))^2} = \frac{h^i\alpha\lambda^{ij}\gamma^i}{(g^i(\Delta^i,\lambda^{ij}))^2} > 0$$
and
$$\frac{\partial\tilde z_f^i(\Delta^i,\lambda^{ij})}{\partial\lambda^{ij}} = \frac{h^i\partial g^i(\Delta^i,\lambda^{ij})/\partial\lambda^{ij}}{(g^i(\Delta^i,\lambda^{ij}))^2} = \frac{h^i\alpha\Delta^i\gamma^i}{(g^i(\Delta^i,\lambda^{ij}))^2} > 0.$$
Define
$$\tilde\lambda_f^{ij}(\Delta^j,z^i) = A_j'^{-1}\left(\alpha\Delta^j\tilde S^i(z^i)\right).$$
Then, for fixed Δ^j, $\tilde\lambda_f^{ij}(\Delta^j,z^i)$ is the λ^{ij}-response function, if this response is less than one. From the inverse function theorem,
$$\frac{\partial\tilde\lambda_f^{ij}(\Delta^j,z^i)}{\partial\Delta^j} = \frac{\alpha\tilde S^i(z^i)}{A_j''(\tilde\lambda_f^{ij}(\Delta^j,z^i))}$$
and
$$\frac{\partial\tilde\lambda_f^{ij}(\Delta^j,z^i)}{\partial z^i} = \frac{\alpha\Delta^j}{A_j''(\tilde\lambda_f^{ij}(\Delta^j,z^i))}\frac{\partial\tilde S^i(z^i)}{\partial z^i} = \frac{-\alpha\Delta^j\gamma^i\Pr(\varepsilon^i>\Phi_i^{-1}(z))}{A_j''(\tilde\lambda_f^{ij}(\Delta^j,z^i))\phi_i(z^i)} < 0.$$
As the partial derivatives of the response functions are of opposite signs there exists a unique solution to equations (40) - (41). Q.E.D.

Proof of Lemma 3

We wish to find bounds on:

$$T^i(\mathbf{\Delta}) = p_2^i(\tilde L^i(z_f^i(\mathbf{\Delta}))-\alpha\Delta^i\lambda_f^{ij}(\mathbf{\Delta})\tilde S^i(z_f^i(\mathbf{\Delta})))+r^ip_1^i+\alpha\Delta^i+p_2^j(A^i(\lambda_f^{ji}(\mathbf{\Delta}))-\alpha\Delta^i\lambda_f^{ji}(\mathbf{\Delta})\tilde S^j(z_f^j(\mathbf{\Delta}))).$$

Note that, by definition of $\lambda_f^{ji}(\mathbf{\Delta})$,

$$\begin{aligned}
&A^i(\lambda_f^{ji}(\mathbf{\Delta}))-\alpha\Delta^i\lambda_f^{ji}(\mathbf{\Delta})\tilde S^j({z_f}^j(\mathbf{\Delta}))\\
\le\ &A^i(\lambda^{ji})-\alpha\Delta^i\lambda^{ji}\tilde S^j({z_f}^j(\mathbf{\Delta}))\Big|_{\lambda^{ji}=0}\\
=\ &-\alpha\Delta^i\tilde S^j({z_f}^j(\mathbf{\Delta}))
\end{aligned}$$

Using $\lambda_f^{ij}(\mathbf{\Delta}),\tilde S^i(\cdot),\tilde S^j(\cdot)\ge 0$ and the definition of z_{my}^i,

$$T^i(\mathbf{\Delta}) \le p_2^i\tilde L^i(z_{my}^i)+r^ip_1^i+\alpha\Delta^i \le p_2^i\tilde L^i(z_{my}^i)+r^ip_1^i+\alpha\Delta_{max}^i$$

where Δ^i_{max} is an upper bound on Δ^i. Thus,

$$\Delta^i_{max} \le p^i_2 \tilde{L}^i(z^i_{my}) + r^i p^i_1 + \alpha \Delta^i_{max}$$

or we can let

$$\Delta^i_{max} = \frac{p^i_2 \tilde{L}^i(z^i_{my}) + r^i p^i_1}{1-\alpha}.$$

Now

$$\begin{aligned} {z_f}^i(\mathbf{\Delta}) &= 1 - \frac{h^i}{\tilde{r}^i + \alpha \Delta^i \gamma^i \lambda^{ij}_f(\mathbf{\Delta}) + h^i} \\ &\le 1 - \frac{h^i}{\tilde{r}^i + \alpha \Delta^i_{max} \gamma^i + h^i}. \end{aligned}$$

Then, for

$$z^i_{max} = 1 - \frac{h^i}{\tilde{r}^i + \alpha \Delta^i_{max} \gamma^i + h^i},$$

${z_f}^i(\mathbf{\Delta}) \le z^i_{max}$. Now,

$$\begin{aligned} \lambda^{ij}_f(\mathbf{\Delta}) &= \min\left(A'^{-1}_j\left(\alpha \Delta^j \tilde{S}^i({z_f}^i(\mathbf{\Delta}))\right), 1\right) \\ &\le \min\left(A'^{-1}_j\left(\alpha \Delta^j_{max} \tilde{S}^i(z^i_{my})\right), 1\right) = \lambda^{ij}_{max} \end{aligned}$$

$$\begin{aligned} T^i(\mathbf{\Delta}) &= p^i_2(\tilde{L}^i({z_f}^i(\mathbf{\Delta})) - \alpha \Delta^i \lambda^{ij}_f(\mathbf{\Delta}) \tilde{S}^i({z_f}^i(\mathbf{\Delta}))) + r^i p^i_1 + \alpha \Delta^i \\ &\quad + p^j_2(A^i(\lambda^{ji}_f(\mathbf{\Delta})) - \alpha \Delta^i \lambda^{ji}_f(\mathbf{\Delta}) \tilde{S}^j({z_f}^j(\mathbf{\Delta}))) \\ &\ge p^i_2(\tilde{L}^i(z^i_{my}) - \alpha \Delta^i \lambda^{ij}_f(\mathbf{\Delta}) \tilde{S}^i(z^i_{my})) + r^i p^i_1 + \alpha \Delta^i \\ &\quad + p^j_2(A^i(\lambda^{ji}_f(\mathbf{\Delta})) - \alpha \Delta^i \lambda^{ji}_f(\mathbf{\Delta}) \tilde{S}^j({z_f}^j(\mathbf{\Delta}))) \\ &\ge p^i_2(\tilde{L}^i(z^i_{my}) - \alpha \Delta^i \lambda^{ij}_{max}) \tilde{S}^i(z^i_{my})) + r^i p^i_1 + \alpha \Delta^i - \alpha p^j_2 \Delta^i \lambda^{ji}_{max} \tilde{S}^j(z^j_{my}) \\ &= p^i_2 \tilde{L}^i(z^i_{my}) + r^i p^i_1 + \alpha \Delta^i(1 - p^i_2 \lambda^{ij}_{max} \tilde{S}^i(z^i_{my}) - p^j_2 \lambda^{ji}_{max} \tilde{S}^j(z^j_{my})) \end{aligned}$$

Thus if $1 - p^i_2 \lambda^{ij}_{max} \tilde{S}^i(z^i_{my}) - p^j_2 \lambda^{ji}_{max} \tilde{S}^j(z^j_{my}) \ge 0$ then let

$$\Delta^i_{min} = \frac{p^i_2 \tilde{L}^i(z^i_{my}) + r^i p^i_1}{1 - p^i_2 \lambda^{ij}_{max} \tilde{S}^i(z^i_{my}) - p^j_2 \lambda^{ji}_{max} \tilde{S}^j(z^j_{my})}$$

else let $\Delta^i_{min} = 0$. Then,

$$\begin{aligned} \lambda^{ij}_f(\mathbf{\Delta}) &= \min\left(A'^{-1}_j\left(\alpha \Delta^j \tilde{S}^i({z_f}^i(\mathbf{\Delta}))\right), 1\right) \\ &\ge \min\left(A'^{-1}_j\left(\alpha \Delta^j_{min} \tilde{S}^i(z^i_{max})\right), 1\right) = \lambda^{ij}_{min} \end{aligned}$$

Q.E.D.

LEMMA 5. *Define,*

$$n^i(\mathbf{\Delta}) = A_j''(\lambda_f^{ij}(\mathbf{\Delta}))\phi_i(z_f^i(\mathbf{\Delta}))(g^i(\Delta^i,\lambda_f^{ij}(\mathbf{\Delta})))^2 + h^i\alpha^2\Delta^i\Delta^j(\gamma^i)^2\Pr(\varepsilon^i > \Phi_i^{-1}(z_f^i(\mathbf{\Delta}))),$$

then

$$\frac{\partial\lambda_f^{ij}(\mathbf{\Delta})}{\partial\Delta^i} = \frac{-h^i\alpha^2\Delta^j(\gamma^i)^2\lambda_f^{ij}(\mathbf{\Delta})\Pr(\varepsilon^i > \Phi_i^{-1}(z_f^i(\mathbf{\Delta})))}{n^i(\mathbf{\Delta})} < 0 \tag{62}$$

$$\frac{\partial\lambda_f^{ij}(\mathbf{\Delta})}{\partial\Delta^j} = \frac{\alpha S^i(z_f^i(\mathbf{\Delta}))\phi_i(z_f^i(\mathbf{\Delta}))(g^i(\Delta^i,\lambda_f^{ij}(\mathbf{\Delta})))^2}{n^i(\mathbf{\Delta})} > 0 \tag{63}$$

$$\frac{\partial z_f^i(\mathbf{\Delta})}{\partial\Delta^i} = \frac{h^i\alpha\gamma^i\lambda_f^{ij}(\mathbf{\Delta})A_j''(\lambda_f^{ij}(\mathbf{\Delta}))}{n^i(\mathbf{\Delta})} > 0 \tag{64}$$

$$\frac{\partial z_f^i(\mathbf{\Delta})}{\partial\Delta^j} = \frac{\alpha^2 h^i\Delta^i\gamma^i S^i(z_f^i(\mathbf{\Delta}))A_j''(\lambda_f^{ij}(\mathbf{\Delta}))}{n^i(\mathbf{\Delta})} > 0 \tag{65}$$

Further,

$$\begin{aligned}\frac{\partial}{\partial\Delta^i}T^i(\mathbf{\Delta}) &= \alpha(1 - p_2^i\lambda_f^{ij}(\mathbf{\Delta})S^i(z_f^i(\mathbf{\Delta})) - p_2^j\lambda_f^{ji}(\mathbf{\Delta})S^j(z_f^j(\mathbf{\Delta}))) \\ &+ \frac{p_2^i h^i\alpha^3\Delta^i\Delta^j(\gamma^i)^2 S^i(z_f^j(\mathbf{\Delta}))\lambda_f^{ij}(\mathbf{\Delta})\Pr(\varepsilon^i > \Phi_i^{-1}(z_f^i(\mathbf{\Delta})))}{n^i(\mathbf{\Delta})} \\ &+ \frac{p_2^j h^j\alpha^3\Delta^i\Delta^j\gamma^j\lambda_f^{ji}(\mathbf{\Delta})\gamma^i S^j(z_f^j(\mathbf{\Delta}))\Pr(\varepsilon^j > \Phi_j^{-1}(z_f^j(\mathbf{\Delta})))A_i''(\lambda_f^{ji}(\mathbf{\Delta}))}{n^j(\mathbf{\Delta})}\end{aligned} \tag{66}$$

$$\begin{aligned}\frac{\partial}{\partial\Delta^j}T^i(\mathbf{\Delta}) &= \frac{-\alpha^2 p_2^i\Delta^i S^i(z_f^i(\mathbf{\Delta}))^2\phi_i(z_f^i(\mathbf{\Delta}))(g^i(\Delta^i,\lambda_f^{ij}(\mathbf{\Delta})))^2}{n^i(\mathbf{\Delta})} \\ &+ \frac{\alpha^2 p_2^j h^j\Delta^i(\gamma^j)^2(\lambda_f^{ji}(\mathbf{\Delta}))^2\Pr(\varepsilon^j > \Phi_j^{-1}(z_f^j(\mathbf{\Delta})))A_i''(\lambda_f^{ji}(\mathbf{\Delta}))}{n^j(\mathbf{\Delta})}.\end{aligned} \tag{67}$$

Proof of Lemma 5

Let us define:

$$G(\Delta^i,\Delta^j,\lambda^{ij}) = \lambda^{ij} - A_j'^{-1}\left(\alpha\Delta^j\tilde{S}^i(\tilde{z}_f^i(\Delta^i,\lambda^{ij}))\right)$$

From the implicit function theorem

$$\frac{\partial\lambda_f^{ij}(\mathbf{\Delta})}{\partial\Delta^i} = \left.\frac{-\frac{\partial G}{\partial\Delta^i}}{\frac{\partial G}{\partial\lambda^{ij}}}\right|_{\lambda^{ij}=\lambda_f^{ij}(\mathbf{\Delta})}.$$

We first compute the appropriate partials as follows.

$$\begin{aligned}\frac{\partial}{\partial\Delta^i}G(\Delta^i,\Delta^j,\lambda^{ij}) &= \frac{-\alpha\Delta^j\left.\frac{\partial\tilde{S}^i(z^i)}{\partial z^i}\right|_{z^i=\tilde{z}_f^i(\Delta^i,\lambda^{ij})}\frac{\partial\tilde{z}_f^i(\Delta^i,\lambda^{ij})}{\partial\Delta^i}}{A_j''(\tilde{\lambda}_f^{ij}(\Delta^j,\tilde{z}_f^i(\Delta^i,\lambda^{ij})))} \\ &= \frac{h^i\alpha^2\Delta^j(\gamma^i)^2\lambda^{ij}\Pr(\varepsilon^i > \Phi_i^{-1}(\tilde{z}_f^i(\Delta^i,\lambda^{ij})))}{A_j''(\tilde{\lambda}_f^{ij}(\Delta^j,\tilde{z}_f^i(\Delta^i,\lambda^{ij})))\phi_i(\tilde{z}_f^i(\Delta^i,\lambda^{ij}))(g^i(\Delta^i,\lambda^{ij}))^2} \\ \frac{\partial}{\partial\Delta^j}G(\Delta^i,\Delta^j,\lambda^{ij}) &= \frac{-\alpha\tilde{S}^i(\tilde{z}_f^i(\Delta^i,\lambda^{ij}))}{A_j''(\tilde{\lambda}_f^{ij}(\Delta^j,\tilde{z}_f^i(\Delta^i,\lambda^{ij})))}\end{aligned}$$

$$\frac{\partial}{\partial \lambda^{ij}} G(\Delta^i, \Delta^j, \lambda^{ij}) = 1 - \frac{\alpha \Delta^j \left.\frac{\partial \tilde{S}^i(z^i)}{\partial z^i}\right|_{z^i = \tilde{z}^i_f(\Delta^i, \lambda^{ij})} \frac{\partial \tilde{z}^i_f(\Delta^i, \lambda^{ij})}{\partial \lambda^{ij}}}{A''_j(\tilde{\lambda}^{ij}_f(\Delta^j, \tilde{z}^i_f(\Delta^i, \lambda^{ij})))}$$
$$= 1 + \frac{h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}(\tilde{z}^i_f(\Delta^i, \lambda^{ij})))}{A''_j(\tilde{\lambda}^{ij}_f(\Delta^j, \tilde{z}^i_f(\Delta^i, \lambda^{ij})))\phi_i(\tilde{z}^i_f(\Delta^i, \lambda^{ij}))(g^i(\Delta^i, \lambda^{ij}))^2}$$

Thus

$$\frac{\partial \lambda^{ij}_f(\mathbf{\Delta})}{\partial \Delta^i}$$
$$= \left.\frac{-\frac{\partial G}{\partial \Delta^i}}{\frac{\partial G}{\partial \lambda^{ij}}}\right|_{\lambda^{ij} = \lambda^{ij}_f(\mathbf{\Delta})}$$
$$= \left.\frac{-h^i \alpha^2 \Delta^j (\gamma^i)^2 \lambda^{ij} \Pr(\varepsilon^i > \Phi_i^{-1}(\tilde{z}^i_f(\Delta^i, \lambda^{ij})))}{A''_j(\tilde{\lambda}^{ij}_f(\Delta^j, \tilde{z}^i_f(\Delta^i, \lambda^{ij})))\phi_i(\tilde{z}^i_f(\Delta^i, \lambda^{ij}))(g^i(\Delta^i, \lambda^{ij}))^2 + h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}(\tilde{z}^i_f(\Delta^i, \lambda^{ij})))}\right|_{\lambda^{ij} = \lambda^{ij}_f(\mathbf{\Delta})}$$
$$= \frac{-h^i \alpha^2 \Delta^j (\gamma^i)^2 \lambda^{ij}_f(\mathbf{\Delta}) \Pr(\varepsilon^i > \Phi_i^{-1}({z_f}^i(\mathbf{\Delta})))}{A''_j(\lambda^{ij}_f(\mathbf{\Delta}))\phi_i({z_f}^i(\mathbf{\Delta}))(g^i(\Delta^i, \lambda^{ij}_f(\mathbf{\Delta})))^2 + h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}({z_f}^i(\mathbf{\Delta})))}$$

and

$$\frac{\partial \lambda^{ij}_f(\mathbf{\Delta})}{\partial \Delta^j}$$
$$= \left.\frac{-\frac{\partial G}{\partial \Delta^j}}{\frac{\partial G}{\partial \lambda^{ij}}}\right|_{\lambda^{ij} = \lambda^{ij}_f(\mathbf{\Delta})}$$
$$= \left.\frac{\alpha \tilde{S}^i(\tilde{z}^i_f(\Delta^i, \lambda^{ij}))\phi_i(\tilde{z}^i_f(\Delta^i, \lambda^{ij}))(g^i(\Delta^i, \lambda^{ij}))^2}{A''_j(\tilde{\lambda}^{ij}_f(\Delta^j, \tilde{z}^i_f(\Delta^i, \lambda^{ij})))\phi_i(\tilde{z}^i_f(\Delta^i, \lambda^{ij}))(g^i(\Delta^i, \lambda^{ij}))^2 + h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}(\tilde{z}^i_f(\Delta^i, \lambda^{ij})))}\right|_{\lambda^{ij} = \lambda^{ij}_f(\mathbf{\Delta})}$$
$$= \frac{\alpha \tilde{S}^i({z_f}^i(\mathbf{\Delta}))\phi_i({z_f}^i(\mathbf{\Delta}))(g^i(\Delta^i, \lambda^{ij}_f(\mathbf{\Delta})))^2}{A''_j(\lambda^{ij}_f(\mathbf{\Delta}))\phi_i({z_f}^i(\mathbf{\Delta}))(g^i(\Delta^i, \lambda^{ij}_f(\mathbf{\Delta})))^2 + h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}({z_f}^i(\mathbf{\Delta})))}$$

Define:

$$H(\Delta^i, \Delta^j, z^j) = z^j - 1 + \frac{h^j}{g^j(\Delta^j, \tilde{\lambda}^{ji}_f(\Delta^i, z^j))}$$

where

$$\tilde{\lambda}^{ji}_f(\Delta^i, z^j) = A_i'^{-1}\left(\alpha \Delta^i \tilde{S}^j(z^j)\right).$$

From the implicit function theorem

$$\frac{\partial {z_f}^j(\mathbf{\Delta})}{\partial \Delta^j} = \left.\frac{-\frac{\partial H}{\partial \Delta^j}}{\frac{\partial H}{\partial z^j}}\right|_{z^j = {z_f}^j(\mathbf{\Delta})}.$$

We first compute the appropriate partials as follows. As previously, from the inverse function theorem,

$$\frac{\partial \tilde{\lambda}^{ji}_f(\Delta^i, z^j)}{\partial \Delta^i} = \frac{\alpha \tilde{S}^j(z^j)}{A''_i(\tilde{\lambda}^{ji}_f(\Delta^i, z^j))}$$

and

$$\frac{\partial \tilde{\lambda}_f^{ji}(\Delta^i, z^j)}{\partial z^j} = \frac{\alpha \Delta^i}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j))} \frac{\partial \tilde{S}^j(z^j)}{\partial z^j} = \frac{-\alpha \Delta^i \gamma^j \Pr(\varepsilon^j > \Phi_j^{-1}(z^j))}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j))}.$$

Recall,

$$g^j(\Delta^j, \lambda^{ji}) = \tilde{r}^j + \alpha \Delta^j \lambda^{ji} \gamma^j + h^j,$$

so that,

$$\begin{aligned}
\frac{\partial}{\partial z^j} g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)) &= \alpha \Delta^j \gamma^j \frac{\partial}{\partial z^j} \tilde{\lambda}_f^{ji}(\Delta^i, z^j) \\
&= \frac{-\alpha^2 \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z^j))}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j))} \\
\frac{\partial}{\partial \Delta^i} g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)) &= \alpha \Delta^j \gamma^j \frac{\partial}{\partial \Delta^i} \tilde{\lambda}_f^{ji}(\Delta^i, z^j) \\
&= \frac{\alpha^2 \Delta^j \gamma^j \tilde{S}^j(z^j)}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j))} \\
\frac{\partial}{\partial \Delta^j} g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)) &= \alpha \gamma^j \tilde{\lambda}_f^{ji}(\Delta^i, z^j)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial}{\partial z^j} H(\Delta^i, \Delta^j, z^j) &= 1 - \frac{h^j \partial g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j))/\partial z^j}{(g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2} \\
&= 1 + \frac{\alpha^2 h^j \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z^j))}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j)) \phi_j(\tilde{z}_f^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))(g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2} \\
\frac{\partial}{\partial \Delta^j} H(\Delta^i, \Delta^j, z^j) &= \frac{-h^j \partial g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j))/\partial \Delta^j}{(g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2} \\
&= \frac{-h^j \alpha \gamma^j \tilde{\lambda}_f^{ji}(\Delta^i, z^j)}{\phi_j(\tilde{z}_f^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))(g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2} \\
\frac{\partial}{\partial \Delta^i} H(\Delta^i, \Delta^j, z^j) &= \frac{-h^j \partial g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j))/\partial \Delta^i}{(g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2} \\
&= \frac{-\alpha^2 h^j \Delta^j \gamma^j \tilde{S}^j(z^j)}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j)) \phi_j(\tilde{z}_f^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))(g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial {z_f}^j(\mathbf{\Delta})}{\partial \Delta^j} \\
&= \left. \frac{-\frac{\partial H}{\partial \Delta^j}}{\frac{\partial H}{\partial z^j}} \right|_{z^j = {z_f}^j(\mathbf{\Delta})} \\
&= \left. \frac{h^j \alpha \gamma^j \tilde{\lambda}_f^{ji}(\Delta^i, z^j) A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j))}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j)) \phi_j(\tilde{z}_f^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j))) g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2 + \alpha^2 h^j \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z^j))} \right|_{z^j = {z_f}^j(\mathbf{\Delta})} \\
&= \frac{h^j \alpha \gamma^j \lambda_f^{ji}(\mathbf{\Delta}) A_i''(\lambda_f^{ji}(\mathbf{\Delta}))}{A_i''(\lambda_f^{ji}(\mathbf{\Delta})) \phi_j({z_f}^j(\mathbf{\Delta}) g^j(\Delta^j, \lambda_f^{ji}(\mathbf{\Delta})))^2 + \alpha^2 h^j \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}({z_f}^j(\mathbf{\Delta})))}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial {z_f}^j(\boldsymbol{\Delta})}{\partial \Delta^i} \\
&= \left.\frac{-\frac{\partial H}{\partial \Delta^i}}{\frac{\partial H}{\partial z^j}}\right|_{z^j={z_f}^j(\boldsymbol{\Delta})} \\
&= \left.\frac{\alpha^2 h^j \Delta^j \gamma^j \tilde{S}^j(z^j) A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j))}{A_i''(\tilde{\lambda}_f^{ji}(\Delta^i, z^j))\phi_j(\tilde{z}_f^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))g^j(\Delta^j, \tilde{\lambda}_f^{ji}(\Delta^i, z^j)))^2 + \alpha^2 h^j \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z^j))}\right|_{z^j={z_f}^j(\boldsymbol{\Delta})} \\
&= \frac{\alpha^2 h^j \Delta^j \gamma^j \tilde{S}^j({z_f}^j(\boldsymbol{\Delta})) A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))}{A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))\phi_j({z_f}^j(\boldsymbol{\Delta}))(g^j(\Delta^j, \lambda_f^{ji}(\boldsymbol{\Delta})))^2 + \alpha^2 h^j \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}({z_f}^j(\boldsymbol{\Delta})))}
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{\partial}{\partial \Delta^i} T^i(\boldsymbol{\Delta}) \\
&= p_2^i \frac{\partial}{\partial z^i}\left[\tilde{L}^i(z^i) - \alpha \Delta^i \lambda_f^{ij}(\boldsymbol{\Delta})\tilde{S}^i(z^i)\right]\Big|_{z^i={z_f}^i(\boldsymbol{\Delta})} \frac{\partial {z_f}^i(\boldsymbol{\Delta})}{\partial \Delta^i} \\
&\quad + p_2^j \frac{\partial}{\partial \lambda^{ji}}\left[A^i(\lambda^{ji}) - \alpha \Delta^i \lambda^{ji} \tilde{S}^j({z_f}^i(\boldsymbol{\Delta}))\right]\Big|_{\lambda^{ji}=\lambda_f^{ji}(\boldsymbol{\Delta})} \frac{\partial \lambda_f^{ji}(\boldsymbol{\Delta})}{\partial \Delta^i} \\
&\quad + \alpha(1 - p_2^i \lambda_f^{ij}(\boldsymbol{\Delta})\tilde{S}^i({z_f}^i(\boldsymbol{\Delta})) - p_2^j \lambda_f^{ji}(\boldsymbol{\Delta})\tilde{S}^j({z_f}^j(\boldsymbol{\Delta}))) \\
&\quad - \alpha p_2^i \Delta^i \tilde{S}^i({z_f}^i(\boldsymbol{\Delta}))\frac{\partial \lambda_f^{ij}(\boldsymbol{\Delta})}{\partial \Delta^i} - \alpha p_2^i \Delta^i \lambda_f^{ji}(\boldsymbol{\Delta})\frac{\partial \tilde{S}^j({z_f}^j(\boldsymbol{\Delta}))}{\partial \Delta^i} \\
&= \alpha(1 - p_2^i \lambda_f^{ij}(\boldsymbol{\Delta})\tilde{S}^i({z_f}^i(\boldsymbol{\Delta})) - p_2^j \lambda_f^{ji}(\boldsymbol{\Delta})\tilde{S}^j({z_f}^j(\boldsymbol{\Delta}))) - \alpha p_2^i \Delta^i \tilde{S}^i({z_f}^i(\boldsymbol{\Delta}))\frac{\partial \lambda_f^{ij}(\boldsymbol{\Delta})}{\partial \Delta^i} \\
&\quad + \alpha p_2^i \Delta^i \lambda_f^{ji}(\boldsymbol{\Delta})\gamma^i \Pr(\varepsilon^j > {z_f}^j(\boldsymbol{\Delta})))\frac{\partial {z_f}^j(\boldsymbol{\Delta})}{\partial \Delta^i} \\
&= \alpha(1 - p_2^i \lambda_f^{ij}(\boldsymbol{\Delta})\tilde{S}^i({z_f}^i(\boldsymbol{\Delta})) - p_2^j \lambda_f^{ji}(\boldsymbol{\Delta})\tilde{S}^j({z_f}^j(\boldsymbol{\Delta}))) \\
&\quad + \frac{p_2^i h^i \alpha^3 \Delta^i \Delta^j (\gamma^i)^2 \tilde{S}^i({z_f}^j(\boldsymbol{\Delta}))\lambda_f^{ij}(\boldsymbol{\Delta}) \Pr(\varepsilon^i > \Phi_i^{-1}({z_f}^i(\boldsymbol{\Delta})))}{A_j''(\lambda_f^{ij}(\boldsymbol{\Delta}))\phi_i({z_f}^i(\boldsymbol{\Delta}))(g^i(\Delta^i, \lambda_f^{ij}(\boldsymbol{\Delta})))^2 + h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}({z_f}^i(\boldsymbol{\Delta})))} \\
&\quad + \frac{p_2^i h^j \alpha^3 \Delta^i \Delta^j \gamma^j \lambda_f^{ji}(\boldsymbol{\Delta})\gamma^i \tilde{S}^j({z_f}^j(\boldsymbol{\Delta})) \Pr(\varepsilon^j > \Phi_j^{-1}({z_f}^j(\boldsymbol{\Delta})))A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))}{A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))\phi_j({z_f}^j(\boldsymbol{\Delta}))(g^j(\Delta^j, \lambda_f^{ji}(\boldsymbol{\Delta})))^2 + \alpha^2 h^j \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}({z_f}^j(\boldsymbol{\Delta})))}
\end{aligned}$$

Thus, $\frac{\partial}{\partial \Delta^i} T^i(\boldsymbol{\Delta}) \geq 0$.

$$\begin{aligned}
&\frac{\partial}{\partial \Delta^j} T^i(\boldsymbol{\Delta}) \\
&= p_2^i \frac{\partial}{\partial z^i}\left[\tilde{L}^i(z^i) - \alpha \Delta^i \lambda_f^{ij}(\boldsymbol{\Delta})\tilde{S}^i(z^i)\right]\Big|_{z^i={z_f}^i(\boldsymbol{\Delta})} \frac{\partial {z_f}^i(\boldsymbol{\Delta})}{\partial \Delta^j} \\
&\quad + p_2^j \frac{\partial}{\partial \lambda^{ji}}\left[A^i(\lambda^{ji}) - \alpha \Delta^i \lambda^{ji} \tilde{S}^j({z_f}^i(\boldsymbol{\Delta}))\right]\Big|_{\lambda^{ji}=\lambda_f^{ji}(\boldsymbol{\Delta})} \frac{\partial \lambda_f^{ji}(\boldsymbol{\Delta})}{\partial \Delta^j} \\
&\quad - \alpha p_2^i \Delta^i \tilde{S}^i({z_f}^j(\boldsymbol{\Delta}))\frac{\partial \lambda_f^{ij}(\boldsymbol{\Delta})}{\partial \Delta^j} - \alpha p_2^j \Delta^i \lambda_f^{ji}(\boldsymbol{\Delta})\frac{\partial \tilde{S}^j({z_f}^j(\boldsymbol{\Delta}))}{\partial \Delta^j} \\
&= -\alpha p_2^i \Delta^i \tilde{S}^i({z_f}^j(\boldsymbol{\Delta}))\frac{\partial \lambda_f^{ij}(\boldsymbol{\Delta})}{\partial \Delta^j} + \alpha p_2^j \Delta^i \lambda_f^{ji}(\boldsymbol{\Delta})\gamma^j \Pr(\varepsilon^j > {z_f}^j(\boldsymbol{\Delta}))\frac{\partial {z_f}^j(\boldsymbol{\Delta})}{\partial \Delta^j} \\
&= \frac{-\alpha^2 p_2^i \Delta^i (\tilde{S}^i({z_f}^i(\boldsymbol{\Delta})))^2 \phi_i({z_f}^i(\boldsymbol{\Delta}))(g^i(\Delta^i, \lambda_f^{ij}(\boldsymbol{\Delta})))^2}{A_j''(\lambda_f^{ij}(\boldsymbol{\Delta}))\phi_i({z_f}^i(\boldsymbol{\Delta}))(g^i(\Delta^i, \lambda_f^{ij}(\boldsymbol{\Delta})))^2 + h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}({z_f}^i(\boldsymbol{\Delta})))}
\end{aligned}$$

$$+\frac{\alpha^2 p_2^j h^j \Delta^i (\gamma^j)^2 (\lambda_f^{ji}(\mathbf{\Delta}))^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z_f{}^j(\mathbf{\Delta})))A_i''(\lambda_f^{ji}(\mathbf{\Delta}))}{A_i''(\lambda_f^{ji}(\mathbf{\Delta}))\phi_j(z_f{}^j(\mathbf{\Delta})g^j(\Delta^j,\lambda_f^{ji}(\mathbf{\Delta})))^2 + \alpha^2 h^j \Delta^i \Delta^j (\gamma^j)^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z_f{}^j(\mathbf{\Delta})))}$$

Substituting in $n^i(\mathbf{\Delta})$ and $n^j(\mathbf{\Delta})$ yields equations (66) and (67). Q.E.D.

Proof of Lemma 4

We wish to show:

$$\left|\frac{\partial}{\partial \Delta^j} T^i(\mathbf{\Delta})\right| < 1$$

which with $|\frac{\partial}{\partial \Delta^i} T^j(\mathbf{\Delta})| < 1$ are the conditions necessary for the model to be a contraction mapping[5]. This follows immediatedly from assumption 13 and Lemma 5.

We now show the result under assumption 12. As the first term of $\frac{\partial}{\partial \Delta^j} T^i(\mathbf{\Delta})$ is negative and the second is positive

$$\left|\frac{\partial}{\partial \Delta^j} T^i(\mathbf{\Delta})\right| \leq \max\left(\frac{\alpha^2 p_2^i \Delta^i (\tilde{S}^i(z_f{}^i(\mathbf{\Delta})))^2 \phi_i(z_f{}^i(\mathbf{\Delta}))(g^i(\Delta^i, \lambda_f^{ij}(\mathbf{\Delta})))^2}{n^i(\mathbf{\Delta})}, \frac{\alpha^2 p_2^j h^j \Delta^i (\gamma^j)^2 (\lambda_f^{ji}(\mathbf{\Delta}))^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z_f{}^j(\mathbf{\Delta})))A_i''(\lambda_f^{ji}(\mathbf{\Delta}))}{n^j(\mathbf{\Delta})}\right)$$

Thus, it suffices to show that,

$$n^i(\mathbf{\Delta}) - \alpha^2 p_2^i \Delta^i (\tilde{S}^i(z_f{}^i(\mathbf{\Delta})))^2 \phi_i(z_f{}^i(\mathbf{\Delta}))(g^i(\Delta^i, \lambda_f^{ij}(\mathbf{\Delta})))^2 > 0 \tag{68}$$

and

$$n^j(\mathbf{\Delta}) - \alpha^2 p_2^j h^j \Delta^i (\gamma^j)^2 (\lambda_f^{ji}(\mathbf{\Delta}))^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z_f{}^j(\mathbf{\Delta})))A_i''(\lambda_f^{ji}(\mathbf{\Delta})) > 0 \tag{69}$$

Recall,

$$n^i(\mathbf{\Delta}) = A_j''(\lambda_f^{ij}(\mathbf{\Delta}))\phi_i(z_f{}^i(\mathbf{\Delta}))(g^i(\Delta^i, \lambda_f^{ij}(\mathbf{\Delta})))^2 + h^i \alpha^2 \Delta^i \Delta^j (\gamma^i)^2 \Pr(\varepsilon^i > \Phi_i^{-1}(z_f{}^i(\mathbf{\Delta}))),$$

Therefore, a sufficient condition for (68) is that

$$\alpha^2 p_2^i \Delta^i (\tilde{S}^i(z_f{}^i(\mathbf{\Delta})))^2 < A_j''(\lambda_f^{ij}(\mathbf{\Delta}))$$

for any vector $\mathbf{\Delta}$. This is guaranteed by assumption 12. Two alternate sufficient conditions for (69) are that

$$\Delta^j > p_2^j (\lambda_f^{ji}(\mathbf{\Delta}))^2 A_i''(\lambda_f^{ji}(\mathbf{\Delta}))$$

or

$$\phi_j(z_f{}^j(\mathbf{\Delta}))(g^j(\Delta^j, \lambda_f^{ji}(\mathbf{\Delta})))^2 > \alpha^2 p_2^j h^j \Delta^i (\gamma^j)^2 (\lambda_f^{ji}(\mathbf{\Delta}))^2 \Pr(\varepsilon^j > \Phi_j^{-1}(z_f{}^j(\mathbf{\Delta}))).$$

[5] These definitions collectively are identical to having a spectral radius less than 1 for our two player game.

But

$$(g^j(\Delta^j, \lambda_f^{ji}(\mathbf{\Delta})))^2 \geq 2h^j \alpha \Delta^j \gamma^j \lambda_f^{ji}(\mathbf{\Delta}))$$

so for the latter condition it suffices to show

$$\phi_j({z_f}^j(\mathbf{\Delta}))2\Delta^j > \alpha p_2^j \Delta^i \gamma^j \lambda_f^{ji}(\mathbf{\Delta}))\Pr(\varepsilon^j > \Phi_j^{-1}({z_f}^j(\mathbf{\Delta}))).$$

Assumption 12 guarantees this. Q.E.D.

Proof of Proposition 2

Taking the contraction mapping for firm i:

$$T^i(\mathbf{\Delta}) = p_2^i(\tilde{L}^i(z_f^i(\mathbf{\Delta})) - \alpha\Delta^i\lambda_f^{ij}(\mathbf{\Delta})\tilde{S}^i(z_f^i(\mathbf{\Delta}))) + r^i p_1^i + \alpha\Delta^i + p_2^j(A^i(\lambda_f^{ji}(\mathbf{\Delta})) - \alpha\Delta^i\lambda_f^{ji}(\mathbf{\Delta})\tilde{S}^j(z_f^j(\mathbf{\Delta})))$$

we can construct the function $G(\mathbf{\Delta}) = \Delta^i - T^i(\mathbf{\Delta})$. From Lemma 5

$$\begin{array}{cccc}\dfrac{\partial z_f^i(\mathbf{\Delta})}{\partial \Delta^i} > 0, & \dfrac{\partial z_f^i(\mathbf{\Delta})}{\partial \Delta^j} > 0, & \dfrac{\partial z_f^j(\mathbf{\Delta})}{\partial \Delta^j} > 0, & \dfrac{\partial z_f^j(\mathbf{\Delta})}{\partial \Delta^i} > 0, \\ \dfrac{\partial \lambda_f^{ij}(\mathbf{\Delta})}{\partial \Delta^i} < 0, & \dfrac{\partial \lambda_f^{ij}(\mathbf{\Delta})}{\partial \Delta^j} > 0, & \dfrac{\partial \lambda_f^{ji}(\mathbf{\Delta})}{\partial \Delta^i} > 0, & \dfrac{\partial \lambda_f^{ji}(\mathbf{\Delta})}{\partial \Delta^j} < 0.\end{array}$$

which are used in multiple locations to establish the signs of various partial differentiations. We differentiate G in preparation for applying the implicit function theorem.

$$\begin{aligned}\frac{\partial}{\partial \Delta^i}G(\mathbf{\Delta}) &= 1 - p_2^i \frac{\partial}{\partial z^i}\left[\tilde{L}(z^i) - \alpha\Delta^i\lambda_f^{ij}(\mathbf{\Delta})\tilde{S}^i(z)\right]\Bigg|_{z^i = z_f^i(\mathbf{\Delta})} \frac{\partial z_f^i(\mathbf{\Delta})}{\partial \Delta^i} \\ &\quad - p_2^j \frac{\partial}{\partial \lambda^{ji}}\left[A^i(\lambda^{ji}) - \alpha\Delta^i\lambda^{ji}\tilde{S}^j(z_f^j(\mathbf{\Delta}))\right]\Bigg|_{\lambda^{ji} = \lambda_f^{ji}(\mathbf{\Delta})} \frac{\partial \lambda_f^{ji}(\mathbf{\Delta})}{\partial \Delta^i} \\ &\quad - \alpha(1 - p_2^i\lambda_f^{ij}(\mathbf{\Delta})\tilde{S}^i(z_f^i(\Delta^*)) - p_2^j\lambda_f^{ji}(\mathbf{\Delta})\tilde{S}^j(z_f^j(\mathbf{\Delta}))) \\ &\quad + \alpha p_2^j\Delta^i\lambda_f^{ji}(\mathbf{\Delta})\frac{\partial \tilde{S}^j(z_f^j(\mathbf{\Delta}))}{\partial z^j}\Bigg|_{z^j = z_f^j(\mathbf{\Delta})} \frac{\partial z_f^j(\mathbf{\Delta})}{\partial \Delta^i} + \alpha p_2^i\Delta^i\tilde{S}^i(z_f^i(\mathbf{\Delta}))\frac{\partial \lambda_f^{ij}(\mathbf{\Delta})}{\partial \Delta^i} \\ &= 1 - \alpha(1 - p_2^i\lambda_f^{ij}(\mathbf{\Delta})\tilde{S}^i(z_f^i(\Delta^*)) - p_2^j\lambda_f^{ji}(\mathbf{\Delta})\tilde{S}^j(z_f^j(\mathbf{\Delta}))) \\ &\quad + \alpha p_2^j\Delta^i\lambda_f^{ji}(\mathbf{\Delta})\frac{\partial \tilde{S}^j(z_f^j(\mathbf{\Delta}))}{\partial z^j}\Bigg|_{z^j = z_f^j(\mathbf{\Delta})} \frac{\partial z_f^j(\mathbf{\Delta})}{\partial \Delta^i} + \alpha p_2^i\Delta^i\tilde{S}^i(z_f^i(\mathbf{\Delta}))\frac{\partial \lambda_f^{ij}(\mathbf{\Delta})}{\partial \Delta^i} \\ &= 1 - \alpha\left(1 - p_2^i\lambda_f^{ij}(\mathbf{\Delta})\tilde{S}^i(z_f^i(\Delta^*))\left(1 - \frac{h^i\alpha\Delta^i\Delta^j(\gamma^i)^2\Pr(\varepsilon^i > \Phi_i^{-1}(z_f^i(\mathbf{\Delta})))}{n^i(\mathbf{\Delta})}\right)\right. \\ &\quad \left. - p_2^j\lambda_f^{ji}(\mathbf{\Delta})\left(\tilde{S}^j(z_f^j(\mathbf{\Delta})) - \Delta^i\frac{\partial \tilde{S}^j(z_f^j(\mathbf{\Delta}))}{\partial \Delta^i}\right)\right) \geq 0\end{aligned}$$

which is true due to the condition in the Proposition statement.

$$\begin{aligned}-\frac{\partial}{\partial r^i}G(\mathbf{\Delta}) &= \frac{\partial}{\partial r^i}T(\mathbf{\Delta}) \\ &= p_2^i\frac{\partial}{\partial r^i}\tilde{L}(z_f^i(\mathbf{\Delta})) + p_1^i - \alpha p_2^j\Delta^i\lambda_f^{ji}(\mathbf{\Delta})\,\frac{\partial \tilde{S}^j(z_f^j(\mathbf{\Delta}))}{\partial z^j}\Bigg|_{z^j = z_f^j(\mathbf{\Delta})} \frac{\partial z_f^j(\mathbf{\Delta})}{\partial r^i}\end{aligned}$$

$$
\begin{aligned}
& -\alpha p_2^i \Delta^i \tilde{S}^i(z_f^i(\boldsymbol{\Delta}))\frac{\partial \lambda_f^{ij}(\boldsymbol{\Delta})}{\partial r^i} \\
&= -p_2^i(1-\alpha(1-\gamma^i))\mathrm{E}[(\varepsilon^i - z_f^i(\boldsymbol{\Delta}))^+] + p_1^i \geq 0 \quad \text{ if } p_1^i \geq p_2^i \text{ since } \frac{dz_f^j}{dr^i} = \frac{d\lambda_f^{ij}}{dr^i} = 0 \\
-\frac{\partial}{\partial r^j}G(\boldsymbol{\Delta}) &= \frac{\partial}{\partial r^j}T(\boldsymbol{\Delta}) \\
&= \alpha p_2^j \Delta^i \lambda_f^{ji}(\boldsymbol{\Delta})\gamma^j \Pr(\varepsilon^j > \Phi_j^{-1}(z_f^j(\boldsymbol{\Delta})))\frac{\partial z_f^j(\boldsymbol{\Delta})}{\partial r^j} \geq 0 \\
-\frac{\partial}{\partial h^i}G(\boldsymbol{\Delta}) &= \frac{\partial}{\partial h^i}T(\boldsymbol{\Delta}) = p_2^i \frac{\partial}{\partial h^i}\tilde{L}^i(z_f^i(\boldsymbol{\Delta})) = -p_2^i \mathrm{E}[(z_f^i(\boldsymbol{\Delta}) - \varepsilon^i)^+] \leq 0 \\
-\frac{\partial}{\partial h^j}G(\boldsymbol{\Delta}) &= \frac{\partial}{\partial h^j}T(\boldsymbol{\Delta}) \\
&= \frac{-\alpha p_2^j \Delta^i \lambda_f^{ji}(\boldsymbol{\Delta})\gamma^j \Pr(\varepsilon^j > \Phi_j^{-1}(z_f^j(\boldsymbol{\Delta})))(\tilde{r}^j + \alpha\Delta^j\lambda_f^{ji}(\boldsymbol{\Delta})\gamma^j)}{\phi_j(z_f^i(\boldsymbol{\Delta}))[g^j(\boldsymbol{\Delta}, \lambda_f^{ji}(\boldsymbol{\Delta}))]^2} \leq 0 \\
-\frac{\partial}{\partial \gamma^i}G(\boldsymbol{\Delta}) &= \frac{\partial}{\partial \gamma^i}T(\boldsymbol{\Delta}) \\
&= p_2^i \frac{\partial}{\partial \gamma^i}\tilde{L}(z_f^i(\boldsymbol{\Delta})) - \alpha p_2^i \Delta^i \mathrm{E}[(\varepsilon^i - z_f^i(\boldsymbol{\Delta}))^+]\lambda_f^{ij}(\boldsymbol{\Delta}) \\
& -p_2^i \alpha \Delta^i \tilde{S}^i(z_f^i(\boldsymbol{\Delta}))\frac{\partial \lambda_f^{ij}(\boldsymbol{\Delta})}{\partial \gamma^i} \\
&= -r^i p_2^i \alpha \mathrm{E}[(\varepsilon^i - z_f^i(\boldsymbol{\Delta}))^+] - \alpha p_2^i \Delta^i \mathrm{E}[(\varepsilon^i - z_f^i(\boldsymbol{\Delta}))^+]\lambda_f^{ij}(\boldsymbol{\Delta}) \\
& -\frac{\alpha p_2^i \Delta^i \tilde{S}^i(z_f^i(\boldsymbol{\Delta}))\alpha\Delta^j \mathrm{E}[(\varepsilon^i - z_f^i(\boldsymbol{\Delta}))^+]}{A_j''(\lambda_f^{ij}(\boldsymbol{\Delta}))} \leq 0
\end{aligned}
$$

Applying the implicit function theorem:

$$
\frac{\partial \Delta^i}{\partial r^i} = \frac{-\frac{\partial G}{\partial r^i}}{\frac{\partial G}{\partial \Delta^i}} \geq 0, \quad \frac{\partial \Delta^i}{\partial r^j} = \frac{-\frac{\partial G}{\partial r^j}}{\frac{\partial G}{\partial \Delta^i}} \geq 0, \quad \frac{\partial \Delta^i}{\partial h^i} = \frac{-\frac{\partial G}{\partial h^i}}{\frac{\partial G}{\partial \Delta^i}} \leq 0, \quad \frac{\partial \Delta^i}{\partial h^j} = \frac{-\frac{\partial G}{\partial h^j}}{\frac{\partial G}{\partial \Delta^i}} \leq 0, \quad \frac{\partial \Delta^i}{\partial \gamma^i} = \frac{-\frac{\partial G}{\partial \gamma^i}}{\frac{\partial G}{\partial \Delta^i}} \leq 0.
$$

And similarly,

$$
\begin{aligned}
\frac{dz_f^i(\boldsymbol{\Delta})}{dr^i} &= \frac{h^i(1-\alpha(1-\gamma^i))}{\phi_i(z_f^i(\boldsymbol{\Delta}))[g^i(\boldsymbol{\Delta}, \lambda_f^{ij}(\boldsymbol{\Delta}))]^2} + \frac{\partial z_f^i(\boldsymbol{\Delta})}{\partial \Delta^i}\frac{\partial \Delta^i}{\partial r^i} \geq 0 \\
\frac{dz_f^i(\boldsymbol{\Delta})}{dr^j} &= \frac{\partial z_f^i(\boldsymbol{\Delta})}{\partial \Delta^i}\frac{\partial \Delta^i}{\partial r^j} \geq 0 \qquad \text{since } \frac{dz^i}{dr^j} = 0 \\
\frac{dz_f^i(\boldsymbol{\Delta})}{dh^i} &= \frac{-(\tilde{r}^i + \alpha\Delta^i\lambda_f^{ij}(\boldsymbol{\Delta})\gamma^i)}{\phi_i(z_f^i(\boldsymbol{\Delta}))[g^i(\boldsymbol{\Delta}, \lambda_f^{ij}(\boldsymbol{\Delta}))]^2} + \frac{\partial z_f^i(\boldsymbol{\Delta})}{\partial \Delta^i}\frac{\partial \Delta^i}{\partial h^i} \leq 0 \\
\frac{dz_f^i(\boldsymbol{\Delta})}{dh^j} &= \frac{\partial z_f^i(\boldsymbol{\Delta})}{\partial \Delta^i}\frac{\partial \Delta^i}{\partial h^j} \leq 0 \qquad \text{since } \frac{dz^i}{dh^j} = 0 \\
\frac{d\lambda_f^{ji}(\boldsymbol{\Delta})}{dr^i} &= \frac{\alpha\tilde{S}^j(z_f^j(\boldsymbol{\Delta}))}{A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))}\frac{\partial \Delta^i}{\partial r^i} \geq 0 \\
\frac{d\lambda_f^{ji}(\boldsymbol{\Delta})}{dr^j} &= \frac{\alpha\tilde{S}^j(z_f^j(\boldsymbol{\Delta}))}{A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))}\frac{\partial \Delta^i}{\partial r^j} \geq 0 \\
\frac{d\lambda_f^{ji}(\boldsymbol{\Delta})}{dh^i} &= \frac{\alpha\tilde{S}^j(z_f^j(\boldsymbol{\Delta}))}{A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))}\frac{\partial \Delta^i}{\partial h^i} \leq 0 \\
\frac{d\lambda_f^{ji}(\boldsymbol{\Delta})}{dh^j} &= \frac{\alpha\tilde{S}^j(z_f^j(\boldsymbol{\Delta}))}{A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))}\frac{\partial \Delta^i}{\partial h^j} \leq 0 \\
\frac{d\lambda_f^{ji}(\boldsymbol{\Delta})}{d\gamma^i} &= \frac{\alpha\tilde{S}^j(z_f^j(\boldsymbol{\Delta}))}{A_i''(\lambda_f^{ji}(\boldsymbol{\Delta}))}\frac{\partial \Delta^i}{\partial \gamma^i} \geq 0
\end{aligned}
$$

Q.E.D.

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