Online Appendix to "On Markov Equilibria in Dynamic Inventory Competition"

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Appendix EC.1: Basestock Equilibrium

Theorem EC.1 below shows that a basestock policy is an ME. However, the condition required to prove it (Property 6) is quite strong. We first require an analog to Definition 5.

Definition EC.1 (Potential Reward for Overstocking). Let $\varepsilon^i_{T+1}(x) = 0$ for all $x \geq 0$. Then recursively define

$$\varepsilon^i_t(x) = \begin{cases} 0 & 0 \leq x \leq y^{i*} \\ \max_{z \leq x} \left\{ \pi^i(z, R^j(z)) - \pi^{i*} + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(z, R^j(z)))\right] \right\} & x > y^{i*}. \end{cases}$$

Thus, $\varepsilon^i_t(x)$ is an upper bound on the extra reward available to firm i for an overstock of x in period t, assuming firm j reacts to that overstock according to the response function $R^j(\cdot)$. Note that by definition $\varepsilon^i_t(\cdot)$ is nondecreasing and $\varepsilon^i_t(y^{i*}) = 0$.

Property 6. For $x^i \geq \min(y^{i*}, R^i(x^j))$, $x^j \geq \min(y^{j*}, R^j(x^i))$, $\pi^i(x^i, x^j) + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(x^i, x^j))\right]$ is nonincreasing in (x^i, x^j). Further, $\pi^i(x^i, R^j(x^i)) + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(x^i, R^j(x^i)))\right]$ is nonincreasing in x^i.

Property 6 implies that $\pi^i(y, y^{j*}) + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, y^{j*}))\right]$ is nonincreasing in y beyond y^{i*}. This is the key assumption for the SSE to remain an equilibrium: it is not worth firm i's while to overstock in this period purely for next period's gain if it does not get an immediate response from firm j. To keep basestock always a best response it has to be assumed at all $x^j \geq \min(y^{j*}, R^j(x^i))$, as in the actual property given above, and further the extra gain needs to be nonincreasing so that no improvement can be made by moving off a high inventory level to an even higher level. Note that considerable effort was applied to prove the nonincreasing nature of the functions in Property 6 under some exogenous conditions, but we found no natural conditions.

Definition EC.2 (Residual Value). Let $\nu^i_{T+1}(x^i, x^j) = 0$ for all $x^i, x^j \geq 0$. Then recursively define

$$\nu^i_t(x^i,x^j) = \begin{cases} 0 & 0 \leq x^i \leq y^{i*},\ 0 \leq x^j \leq y^{j*} \\ \pi^i(R^i(x^j), x^j) - \pi^{i*} + \alpha \mathrm{E}\left[\nu^i_{t+1}(X(R^i(x^j), x^j))\right] & x^i \leq R^i(x^j) \leq y^{i*},\ x^j > y^{j*} \\ \pi^i(x^i, R^j(x^i)) + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(x^i, R^j(x^i)))\right] + \alpha \mathrm{E}\left[\nu^i_{t+1}(X(x^i, R^j(x^i)))\right] - \varepsilon^i_t(x^i) - \pi^{i*} & x^i > y^{i*},\ x^j \leq R^j(x^i) \\ \pi^i(x^i, x^j) + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(x^i, x^j))\right] + \alpha \mathrm{E}\left[\nu^i_{t+1}(X(x^i, x^j))\right] - \varepsilon^i_t(x^i) - \pi^{i*} & x^i > R^i(x^j),\ x^j > R^j(x^i). \end{cases}$$

We will show in Theorem EC.1 that $V^i_t(x^i, x^j) = V^{i*} + \varepsilon^i_t(x^i) + \nu^i_t(x^i, x^j)$ and hence $\nu^i_t(\cdot)$ is the residual value for the value-to-go function beyond V^{i*} and $\varepsilon^i_t(\cdot)$. The solutions to the various cases Theorem EC.1's proof can be seen in Figure 1 from Section 2, which also correspond to the four cases for the range in Definition EC.2.

Lemma EC.1. *Under Properties 1, 2, 3, and 6, function $\nu^i_t(x^i, x^j)$ is constant in x^i for $x^i \leq \min(R^i(x^j), y^{i*})$, is constant in x^j for $x^j \leq \min(R^j(x^i), y^{j*})$, and is nonincreasing in both x^i and x^j.*

Proof The proof is by induction. Assume that $\nu^i_{t+1}(x^i, x^j)$ satisfies the given properties. We examine the properties of $\nu^i_t(x^i, x^j)$ from its definition line by line.

Line 1 ($0 \leq x^i \leq y^{i*},\ 0 \leq x^j \leq y^{j*}$): The function is trivially both constant and nonincreasing.

Line 2 ($x^i \leq R^i(x^j) \leq y^{i*},\ x^j > y^{j*}$): The function is constant in x^i. For the first term, $R^i(x^j)$ is nonincreasing in x^j (Property 2(b)); hence, by Property 1 the first term is nonincreasing in x^j. For the last term, $X^i(R^i(y^j), y^j) \leq R^i(y^j) \leq y^{i*}$, and $X^j(y^j, R^i(y^j))$ is nondecreasing in y^j (by Property 3(c)) so by the properties of $\nu^i_{t+1}(\cdot)$ this term must be nonincreasing in x^j .

Line 3 ($x^i > y^{i*},\ x^j \leq R^j(x^i)$): The sum of the first two terms is nonincreasing in x^i by Property 6. For the third term, $X^j(y^j, R^i(y^j)) \leq y^j$ (by Property 3(a)) and so $R^i(X^j(y^j, R^i(y^j))) \geq R^i(y^j)$ (by Property 2(b)), hence $\nu^i_{t+1}(x, X^j(y^j, R^i(y^j))$ is constant in x for $x \leq \min(y^{i*}, R^i(y^j))$. Further, $X^i(R^i(y^j), y^j) \leq R^i(y^j) \leq y^{i*}$, and $X^j(y^j, R^i(y^j))$ is stochastically nondecreasing in y^j (by Properties 2 and 3) so by the of properties $\nu^i_{t+1}(\cdot)$ this term must be decreasing in x^j.

Line 4 ($x^i > R^i(x^j),\ x^j > R^j(x^i)$): The sum of the first and second term is nonincreasing by Property 6. The third and fourth term follow by the properties of $\nu^i_t(\cdot)$, $X(\cdot)$, and $\varepsilon^i_t(\cdot)$. Q.E.D.

Theorem EC.1. *Under Properties 1, 2, 3 and Property 6, the SSE basestock policy with response functions $R^i(\cdot)$ is a Markov Equilibrium with $V^i_t(x^i, x^j) = V^{i*} + \varepsilon^i_t(x^i) + \nu^i_t(x^i, x^j)$.*

Proof We proceed by induction. Suppose that the SSE policy is Markov from period $t+1$ onwards and $V^i_{t+1}(x^i, x^j) = V^{i*} + \varepsilon^i_{t+1}(x^i) + \nu^i_{t+1}(x^i, x^j)$. Further, suppose in period t firm j follows the SSE basestock response policy. It suffices to show that firm i's optimal response is the SSE basestock response and that $V^i_t(x^i, x^j) = V^{i*} + \varepsilon^i_t(x^i) + \nu^i_t(x^i, x^j)$. The following four cases correspond to the four reference labels in Figure 1.

Case 1: $0 \leq x^i \leq y^{i*},\ 0 \leq x^j \leq y^{j*}$.

$$V^i_t(x^i, x^j) = \max_{y \geq x^i} \left\{ \pi^i(y, y^{j*}) + \alpha \left(\pi^{i*}/(1-\alpha) + \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, y^{j*}))\right] + \mathrm{E}\left[\nu^i_{t+1}(X(y, y^{j*}))\right] \right) \right\}.$$

For $y < y^{i*}$, $\pi^i(y, y^{j*})$ is increasing in y. Further, $X^i(y, y^{j*}) \le y < y^{i*}$ and $X^j(y, y^{j*}) \le y^{j*}$ so the third and fourth term are zero. Hence the equilibrium value is at least y^{i*}. Now, by Property 6, $\pi^i(y, y^{j*}) + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, y^{j*}))\right]$ is maximized at $y = y^{i*}$. Further, $X(y, y^{j*})$ is stochastically non-decreasing in y and $\nu(\cdot)$ is a nonincreasing function, so y^{i*} must be a best response and

$$V^i_t(x^i, x^j) = \pi^i(y^{i*}, y^{j*}) + \alpha\pi^{i*}/(1-\alpha) + 0 + 0 = V^{i*}.$$

Case 2: $x^i \le R^i(x^j) \le y^{i*}$, $x^j > y^{j*}$.

$$V^i_t(x^i, x^j) = \max_{y \ge x^i}\left\{\pi^i(y, x^j) + \alpha\left(\pi^{i*}/(1-\alpha) + \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, x^j))\right] + \mathrm{E}\left[\nu^i_{t+1}(X(y, x^j))\right]\right)\right\}.$$

For $y < R^i(x^j)$, $\pi^i(y, x^j)$ is increasing in y. Further, $X^i(y, x^j) \le y < R^i(x^j) \le y^{i*}$ so the third term is zero. For the fourth term $X(y, x^j)$ is nondecreasing in y and $\nu(\cdot)$ is a nonincreasing function, hence the equilibrium value is at least $R^i(x^j)$. Now, by Property 6, $\pi^i(y, x^j) + \alpha \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, x^j))\right]$ is nonincreasing for $y \ge R^i(x^j)$. As before, the last term is nonincreasing in y, so $R^i(x^j)$ must be a best response. Further,

$$\begin{aligned} V^i_t(x^i, x^j) &= \pi^i(R^i(x^j), x^j) + \alpha\left(\pi^{i*}/(1-\alpha) + \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(R^i(x^j), x^j))\right] + \mathrm{E}\left[\nu^i_{t+1}(X(R^i(x^j), x^j))\right]\right) \\ &= V^{i*} - \pi^{i*} + \pi^i(R^i(x^j), x^j) + \alpha\mathrm{E}\left[\nu^i_{t+1}(X(R^i(x^j), x^j))\right] \\ &= V^{i*} + \nu^t_i(x^i, x^j). \end{aligned}$$

Case 3: $x^i > y^{i*}$, $x^j \le R^j(x^i)$.

$$V^i_t(x^i, x^j) = \max_{y \ge x^i}\left\{\pi^i(y, R^j(x^i)) + \alpha\left(\pi^{i*}/(1-\alpha) + \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, R^j(x^i)))\right] + \mathrm{E}\left[\nu^i_{t+1}(X(y, R^j(x^i)))\right]\right)\right\}.$$

By Property 6, $\pi^i(y, R^j(x^i)) + \alpha\mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, R^j(x^i)))\right]$ is nonincreasing in y for $y > x^i > y^{i*}$. But $X(y, R^j(x^i))$ is stochastically nondecreasing in y and $\nu(\cdot)$ is a nonincreasing function, so x^i must be a best response. Further,

$$\begin{aligned} V^i_t(x^i, x^j) &= \pi^i(x^i, R^j(x^i)) + \alpha\left(\pi^{i*}/(1-\alpha) + \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(x^i, R^j(x^i)))\right] + \mathrm{E}\left[\nu^i_{t+1}(X(x^i, R^j(x^i)))\right]\right) \\ &= V^{i*} - \pi^{i*} + \pi^i(x^i, R^j(x^i)) + \alpha\mathrm{E}\left[\varepsilon^i_{t+1}(X^i(x^i, R^j(x^i)))\right] + \alpha\mathrm{E}\left[\nu^i_{t+1}(X(x^i, R^j(x^i)))\right] \\ &= V^{i*} + \varepsilon^i_t(x^i) + \nu^t_i(x^i, x^j). \end{aligned}$$

Case 4: $x^i > R^i(x^j)$, $x^j > R^j(x^i)$.

$$V^i_t(x^i, x^j) = \max_{y \ge x^i}\left\{\pi^i(y, x^j) + \alpha\left(\pi^{i*}/(1-\alpha) + \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, x^j))\right] + \mathrm{E}\left[\nu^i_{t+1}(X(y, x^j))\right]\right)\right\}.$$

By Property 6, $\pi^i(y, x^j) + \alpha\mathrm{E}\left[\varepsilon^i_{t+1}(X^i(y, x^j))\right]$ is nonincreasing in y for $y > R^i(x^j)$, $X(y, x^j)$ is stochastically nondecreasing in y, and $\nu(\cdot)$ is a nonincreasing function, so x^i must be a best response. Further,

$$\begin{aligned} V^i_t(x^i, x^j) &= \pi^i(x^i, x^j) + \alpha\left(\pi^{i*}/(1-\alpha) + \mathrm{E}\left[\varepsilon^i_{t+1}(X^i(x^i, x^j))\right] + \mathrm{E}\left[\nu^i_{t+1}(X(x^i, x^j))\right]\right) \\ &= V^{i*} + \varepsilon^i_t(x^i) + \nu^t_i(x^i, x^j). \end{aligned}$$

Q.E.D.

Appendix EC.2: Proofs of Lemmas for Application Models of Section 4

Proof of Lemma 3 Taking partial derivatives of $\pi^i(x,z)$,

$$\frac{\partial \pi^i(x,z)}{\partial x} = (r^i - c^i) - (r^i + h^i - \alpha c^i)\mathrm{P}(x > D^i + \gamma^{ji}(D^j - z)^+) \tag{EC.1}$$

$$\frac{\partial \pi^i(x,z)}{\partial z} = -(r^i + h^i - \alpha c^i)\gamma^{ji}\mathrm{P}(D^j - (x - D^i)/\gamma^{ji} < z \le D^j) \tag{EC.2}$$

$$\begin{aligned}\frac{\partial^2 \pi^i(x,z)}{\partial x^2} &= -(r^i + h^i - \alpha c^i) f_{\Lambda^i(z)}(x) \qquad\qquad \text{(EC.3)}\\ &= -(r^i + h^i - \alpha c^i)\left[f_{D^i}(x)\mathrm{P}(D^j \le z) + \int_z^{z+x/\gamma^{ji}} f_{D^i}(x - \gamma^{ji}(w - z)) f_{D^j}(w)dw \right]\end{aligned}$$

$$\frac{\partial^2 \pi^i(x,z)}{\partial x \partial z} = -(r^i + h^i - \alpha c^i)\gamma^{ji} \int_z^{z+x/\gamma^{ji}} f_{D^i}(x - \gamma^{ji}(w - z)) f_{D^j}(w)dw. \tag{EC.4}$$

Equations (EC.4), (EC.3), and (EC.2) imply that $\pi^i(y^i, y^j)$ is (a) submodular in (y^i, y^j), (b) concave in y^i, and (c) nonincreasing in y^j respectively. Q.E.D.

Proof of Lemma 4 Define $K^i(x,z) = \frac{\partial \pi^i(x,z)}{\partial x} = 0$ where

$$\begin{aligned}\frac{\partial \pi^i(x,z)}{\partial x} &= (r^i - c^i) - (r^i + h^i - \alpha c^i)\mathrm{P}(x > D^i + \gamma^{ji}(D^j - z)^+)\\ &= (r^i - c^i) - (r^i + h^i - \alpha c^i)\Big[\mathrm{P}(x > D^i)\mathrm{P}(z > D^j)\\ &\qquad + \int_z^{z+x/\gamma^{ji}} \mathrm{P}(x > D^i + \gamma^{ji}(w - z)) f_{D^j}(w)dw\Big].\end{aligned}$$

We find the derivative of the best response function via the implicit function theorem as follows:

$$\begin{aligned}\frac{\partial R^i(z)}{\partial z} &= \left.\frac{-\frac{\partial K^i}{\partial z}}{\frac{\partial K^i}{\partial x}}\right|_{(R^i(z),z)} \quad \text{where}\\ \frac{\partial K^i}{\partial x} &= -(r^i + h^i - \alpha c^i)\left[f_{D^i}(x)\mathrm{P}(z > D^j) + \int_z^{z+x/\gamma^{ji}} f_{D^i}(x - \gamma^{ji}(w - z)) f_{D^j}(w)dw \right], \text{ and}\\ \frac{\partial K^i}{\partial z} &= -(r^i + h^i - \alpha c^i)\left[\int_z^{z+x/\gamma^{ji}} \gamma^{ji} f_{D^i}(x - \gamma^{ji}(w - z)) f_{D^j}(w)dw \right].\end{aligned}$$

The first thing to note is that the best response function's derivative has a consistent sign (specifically, negative), indicating monotonicity of the function as in (a). Clearly, $\left|\frac{\partial K}{\partial z}\right| < \left|\frac{\partial K}{\partial x}\right|$ because $\gamma^{ji} < 1$ and $f_{D^i}(x)\mathrm{P}(z > D^j) \ge 0$.[9] The result of this is that the magnitude of the slope of the best response function will be less than 1, yielding (b). Q.E.D.

[9] To accommodate $\gamma^{ji} = 1$ one would simply need a sufficient condition on the support of demand to guarantee that $f_{D^i}(x)\mathrm{P}(z > D^j) > 0$.

Proof of Lemma 5 Note

$$X^i(y^i, R^j(y^i)) = \left(y^i - D^i - \gamma^{ji}(D^j - R^j(y^i))^+\right)^+.$$

For any fixed (D^i, D^j), a sufficient condition for this to be nondecreasing is $\gamma^{ji}\frac{\partial R^j(y^i)}{\partial y^i} \geq -1$. But this is true by Lemma 4 and because $0 \leq \gamma^{ji} < 1$. Thus, unconditioning on demand, we have the desired result. Finally, $X^i(\cdot)$ is clearly stochastically nondecreasing in both y^i and y^j so $X(y^i, y^j)$ is stochastically nondecreasing in (y^i, y^j). Q.E.D.

Proof of Lemma 6 Taking partial derivatives of $\pi^i(x,z)$,

$$\frac{\partial \pi^i(x,z)}{\partial x} = (r^i - c^i) - (r^i + h^i - \alpha c^i)(1-\beta_{ii})\mathrm{P}(D^i \leq (1-\beta_{ii})x + \beta_{ij}z) \tag{EC.5}$$

$$\frac{\partial \pi^i(x,z)}{\partial z} = -(r^i + h^i - \alpha c^i)\beta_{ij}\mathrm{P}(D^i \leq (1-\beta_{ii})x + \beta_{ij}z) \tag{EC.6}$$

$$\frac{\partial^2 \pi^i(x,z)}{\partial x^2} = -(r^i + h^i - \alpha c^i)(1-\beta_{ii})^2 f_{D^i}((1-\beta_{ii})x + \beta_{ij}z) \tag{EC.7}$$

$$\frac{\partial^2 \pi^i(x,z)}{\partial x \partial z} = -(r^i + h^i - \alpha c^i)\beta_{ij}(1-\beta_{ii}) f_{D^i}((1-\beta_{ii})x + \beta_{ij}z). \tag{EC.8}$$

Equations (EC.8), (EC.7), and (EC.6) imply that $\pi^i(y^i, y^j)$ is (a) submodular in (y^i, y^j), (b) concave in y^i, and (c) nonincreasing in y^j respectively. Q.E.D.

Proof of Lemma 7 Define $K^i(x,z) = \frac{\partial \pi^i(x,z)}{\partial x} = 0$ where

$$\frac{\partial \pi^i(x,z)}{\partial x} = (r^i - c^i) - (r^i + h^i - \alpha c^i)(1-\beta_{ii})\mathrm{P}(D^i \leq (1-\beta_{ii})x + \beta_{ij}z).$$

We find the derivative of the best response function via the implicit function theorem as follows:

$$\begin{aligned}
\frac{\partial R^i(z)}{\partial z} &= \left.\frac{-\frac{\partial K^i}{\partial z}}{\frac{\partial K^i}{\partial x}}\right|_{(R^i(z),z)} \quad \text{where} \\
\frac{\partial K^i}{\partial x} &= -(r^i + h^i - \alpha c^i)(1-\beta_{ii})^2 f_{D^i}((1-\beta_{ii})x + \beta_{ij}z), \text{ and} \\
\frac{\partial K^i}{\partial z} &= -(r^i + h^i - \alpha c^i)\beta_{ij}(1-\beta_{ii}) f_{D^i}((1-\beta_{ii})x + \beta_{ij}z).
\end{aligned}$$

Thus,

$$\frac{\partial R^i(z)}{\partial z} = \frac{-\beta_{ij}}{1-\beta_{ii}} \leq 0.$$

Clearly, the best response function's derivative has a consistent sign (specifically, negative) since $\beta_{ii} \geq 0$ and $\beta_{ij} \geq 0$. Under the assumption that $\beta_{ii} + \beta_{ij} < 1$, the magnitude of the slope of the best response function will be less than 1 ($\left|\frac{\partial R^i(z)}{\partial z}\right| < 1$), yielding (b). Q.E.D.

Proof of Lemma 8 Note

$$X^i(y^i, R^j(y^i)) = \left(y^i - D^i - \beta_{ii} y^i + \beta_{ij} R^j(y^i)\right)^+.$$

$$\frac{d}{dx} X^i(x, R^j(x)) = \left((1-\beta_{ii}) + \beta_{ij} \frac{\partial R^j(x)}{\partial x}\right) \mathrm{P}(D^i \leq (1-\beta_{ii})x + \beta_{ij} R^j(x)).$$

Since $\beta_{jj} + \beta_{ji} < 1$, $\frac{\partial R^j(x)}{\partial x} < 1$, which with $\beta_{ii} + \beta_{ij} < 1$, is a sufficient condition for $(1-\beta_{ii}) + \beta_{ij} \frac{\partial R^j(x)}{\partial x} > 0$ ensuring $X^i(y^i, R^j(y^i))$ is nondecreasing.

Clearly, $X^i(\cdot)$ is stochastically nondecreasing in both y^i and y^j so $X(y^i, y^j)$ is stochastically nondecreasing in (y^i, y^j). Likewise since $X^i(y^i, y^j) = ((1-\beta_{ii})y^i - D^i + \beta_{ij} y^j)^+ = (y^i - \Lambda(y^i, y^j))^+ \leq y^i$ since $\Lambda(y^i, y^j) \geq 0$. Q.E.D.

Proof of Lemma 9 Taking partial derivatives of $\pi^i(x,z)$,

$$\frac{\partial \pi^i(x,z)}{\partial x} = r^i \frac{\partial \mu^i(x,z)}{\partial x} - (1-\alpha\theta) = r^i \gamma x^{\gamma-1} z^{-\beta} - (1-\theta\alpha) \tag{EC.9}$$

$$\frac{\partial \pi^i(x,z)}{\partial z} = r^i \frac{\partial \mu^i(x,z)}{\partial z} = -r^i \beta x^{\gamma} z^{-\beta-1} \tag{EC.10}$$

$$\frac{\partial^2 \pi^i(x,z)}{\partial x^2} = r^i \frac{\partial^2 \mu^i(x,z)}{\partial x^2} = r^i \gamma(\gamma-1) x^{\gamma-2} z^{-\beta} \tag{EC.11}$$

$$\frac{\partial^2 \pi^i(x,z)}{\partial x \partial z} = r^i \frac{\partial^2 \mu^i(x,z)}{\partial x \partial z} = -r^i \gamma \beta x^{\gamma-1} z^{-\beta-1}. \tag{EC.12}$$

Equations (EC.12), (EC.11), and (EC.10) imply that $\pi^i(y^i, y^j)$ is (a) submodular in (y^i, y^j), (b) concave in y^i, and (c) nonincreasing in y^j. Q.E.D.

Proof of Lemma 10 Note that $R^i(z)$ is the solution (for x) to the equation $\frac{\partial \pi^i(x,z)}{\partial x} = 0$. Therefore, rearranging equation (EC.9) in Lemma 3

$$R^i(z) = \left(\frac{r^i \gamma}{1-\alpha\theta}\right)^{\frac{1}{1-\gamma}} z^{\frac{-\beta}{1-\gamma}}.$$

Thus,

$$\frac{\partial R^i(z)}{\partial z} = \frac{-\beta}{1-\gamma} \left(\frac{r^i \gamma}{1-\alpha\theta}\right)^{\frac{1}{1-\gamma}} z^{\frac{-\beta}{1-\gamma}-1}.$$

Thus, the best response function's derivative has a consistent sign (specifically, negative) and due to the assumptions above has magnitude less than one. Q.E.D.

Proof of Lemma 11 Note

$$X^i(y^i, y^j) = X^i(y^i, R^j(y^i)) = \theta y^i$$

which clearly satisfies all three properties. Q.E.D.

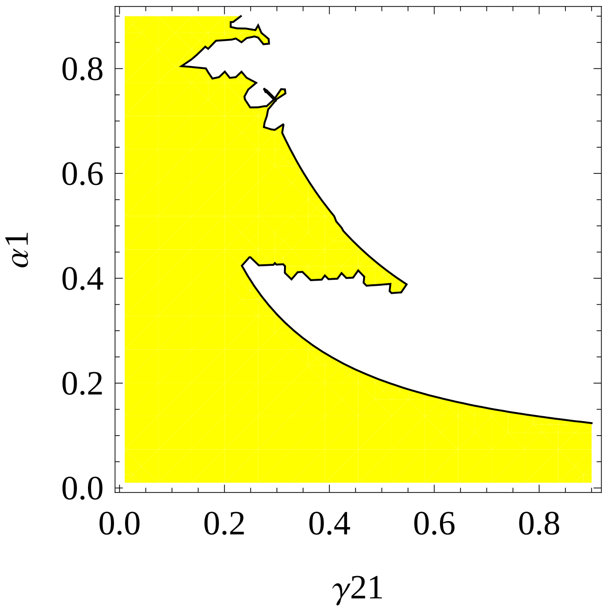

Figure EC.1 Contour plot for Property 4 as a function of α (for player 1) and γ^{21}.

Appendix EC.3: Exploring the Generality of Property 4

Two key properties included in the paper are Properties 4 and 5. As discussed, Property 5 is stronger than Property 4 but has the dual advantage of not requiring a recursive calculation and endowing the reader with intuition. To illustrate the efficacy of Property 4, we have numerically exercised the stockout-based substitution inventory model with uncertain demands distributed as Uniform[0,1] using Mathematica code. Although the calculation of $e_t^i(x)$ described in Definition 5 requires a recursion, the term $e_{t+1}^i(\hat{x}_{t+1}^i)$ is the maximum value in the following period, so the calculation is not overly arduous. That said, practical closed-form expressions cannot be derived, unfortunately, despite using uniformly distributed demands, which suggests that even more elaborate demand distributions would not yield them either. However, we can numerically illustrate circumstances under which Property 4 holds and does not hold. First, taking a somewhat arbitrary example where $(r^i - c^i) = 100$, $r^i + h^i = 150$, $c^i = 20$, $\gamma^{12} = 0.1$, and $\alpha^2 = 0.9$, we can examine Figure EC.1.

The shaded area in Figure EC.1 represents the values of γ^{21} and α^1 where Property 4 holds, whereas the non-shaded area is where the property is violated. The boundary between the regions is not very smooth, which could be partly due to the machine precision but also possibly due to the non-linear dependencies of the profit functions and equilibrium solution (across multiple periods) upon γ^{21} and α^1. We note we can easily find regions where Property 4 is upheld or violated. So it is not difficult to find examples where the property holds and, consequently, where Theorem 2 is true. Likewise, Figure EC.1 also suggests that the SSE may not be Markov everywhere since the property does not hold universally (of course, Property 4 is a sufficient condition so a violation of it doesn't necessarily indicate the SSE is not an ME).

We can vary some of the parameters to see the effect upon Property 4. By reducing the holding cost or unit acquisition cost for firm 1, we see the area where the property is violated expands. The reasoning for this is that it is providing firm 1 with a greater incentive to overstock since it is cheaper for him to do so. Exercising other parameters seem to have either unexpected or non-monotone effects upon the proportions of the shaded and non-shaded regions in figures analogous to Figure EC.1, but we are left with the impression that Property 4 is relatively straightforward to satisfy for a variety of parameter combinations but not universally supported indicating that not all SSE will be Markov, accentuating our message that the SSE concept may be lacking sometimes.

Using the same technique to find numerical examples where Property 5 holds is far more challenging. However, it is possible with other demand distributions (particularly ones that are bounded away from zero) it may hold. We have maintained it in the paper due to its advantages given above but caution the reader that it appears to be quite a strong condition.