Online Companion for

"Optimal Policies for a Capacitated Two-Echelon Inventory System"

Operations Research
Volume 52, Number 5
September-October 2004

Roman Kapuscinski
University of Michigan

Rodney P. Parker
Yale University

Lemma A1 *Consider two functions, J^A and J^B, each jointly convex and separable in their variables, Y^1 and Y^2, which satisfy $\frac{\partial}{\partial Y^1} J^B(\tilde{Y}) \leq \frac{\partial}{\partial Y^1} J^A(\tilde{Y})$ and $\frac{\partial}{\partial Y^2} J^B(\tilde{Y}) \leq \frac{\partial}{\partial Y^2} J^A(\tilde{Y})$. If $K^B < K^A$, then $z^{1A} \leq z^{1B}$.*

Proof Let us define ($i \in \{A, B\}$): $\zeta^i := \arg\min_{Y^1 \in S^i} J^i(\tilde{Y})$ and $z^{2i} := \arg\min_{Y^2 \in S^i} J^i(\tilde{Y})$ where $S^i = \{\tilde{X} \subset \Re^2 | Y^2 - Y^1 \leq K^i\}$. If $z^{2i} - \zeta^i \geq K^i, z^{1i} = \arg\min_{Y^1} J^i(Y^1, Y^1 + K^i)$; otherwise $z^{1i} = \zeta^i$.

There are four immediate cases: (i) $z^{2A} - \zeta^A \leq K^A, z^{2B} - \zeta^B \leq K^B$; (ii) $z^{2A} - \zeta^A > K^A, z^{2B} - \zeta^B \leq K^B$; (iii) $z^{2A} - \zeta^A \leq K^A, z^{2B} - \zeta^B > K^B$; and (iv) $z^{2A} - \zeta^A > K^A, z^{2B} - \zeta^B > K^B$. When $z^{2i} - \zeta^i > K^i$, since $z^{1i} = \arg\min_{Y^1} J^i(Y^1, Y^1 + K^i)$, based on convexity of each of the components $\zeta^i \leq z^{1i} \leq z^{2i} - K^i$ (otherwise moving towards interval $(\zeta^i, z^{2i} - K^i)$ would decrease cost). From $\frac{\partial}{\partial Y^1} J^B(\tilde{Y}) \leq \frac{\partial}{\partial Y^1} J^A(\tilde{Y})$ and convexity, $z^{1A} \leq z^{1B}$ in cases (i) and (iii). For case (ii) we note the following facts: $z^{2B} - z^{1B} \leq K^B$ and $\zeta^A \leq z^{1A} \leq z^{2A} - K^A$. Then get $z^{1B} \geq z^{2B} - K^B \geq z^{2A} - K^B \geq z^{2A} - K^A \geq z^{1A}$ since $\frac{\partial}{\partial Y^2} J^B(\tilde{Y}) \leq \frac{\partial}{\partial Y^2} J^A(\tilde{Y})$. For case (iv), we know that $z^{1i} = \arg\min_{Y^1} J^i(Y^1, Y^1 + K^i)$ for $i \in \{A, B\}$. We consider the slopes along the lines $(Y^1, Y^1 + K^i)$. We thus attain the following logic from the separability of the functions:

$$
\begin{aligned}
\frac{\partial}{\partial Y^1} J^A(Y^1, Y^1 + K^A) &= \frac{\partial}{\partial X^1} J^A(\tilde{X})|_{X^1 = Y^1} + \frac{\partial}{\partial X^2} J^A(\tilde{X})|_{X^2 = Y^1 + K^A} \\
&\geq \frac{\partial}{\partial X^1} J^A(\tilde{X})|_{X^1 = Y^1} + \frac{\partial}{\partial X^2} J^A(\tilde{X})|_{X^2 = Y^1 + K^B} \\
&\geq \frac{\partial}{\partial X^1} J^B(\tilde{X})|_{X^1 = Y^1} + \frac{\partial}{\partial X^2} J^B(\tilde{X})|_{X^2 = Y^1 + K^B} \\
&= \frac{\partial}{\partial Y^1} J^B(Y^1, Y^1 + K^B)
\end{aligned}
$$

which implies that $z^{1A} \leq z^{1B}$. ■

Theorem 3 *The optimal base-stock levels, z_n^1 and z_n^2, are*

(a) non-decreasing in period n; and

(b) increasing in K_1 so long as $K_1 \leq K_2$ is maintained.

Proof (a) This proof will demonstrate a dominance of the functional derivatives as follows:

$$
\frac{\partial}{\partial Y^1} J_{n+1}(\tilde{Y}) \leq \frac{\partial}{\partial Y^1} J_n(\tilde{Y}) \text{ for } Y^1 \leq z_n^1, Y^2 \leq \max(z_n^1, z_n^2) \tag{1}
$$

$$
\frac{\partial}{\partial Y^2} J_{n+1}(\tilde{Y}) \leq \frac{\partial}{\partial Y^2} J_n(\tilde{Y}) \text{ for } Y^1 \leq z_n^1, Y^2 \leq \max(z_n^1, z_n^2) \tag{2}
$$

$$\frac{\partial}{\partial X^1}V_n(\tilde{X}) \leq \frac{\partial}{\partial X^1}V_{n-1}(\tilde{X}) \text{ for } X^1 \leq z_n^1, X^2 \leq \max(z_n^1, z_n^2) \tag{3}$$

$$\frac{\partial}{\partial X^2}V_n(\tilde{X}) \leq \frac{\partial}{\partial X^2}V_{n-1}(\tilde{X}) \text{ for } X^1 \leq z_n^1, X^2 \leq \max(z_n^1, z_n^2) \tag{4}$$

In period 1, since the optimal decision for installation 2 is to order nothing, this corresponds to $z_1^2 = -\infty$, and we check for the conditions (1)-(4) for the territory below (z_1^1, z_1^1). We have $\frac{\partial}{\partial X^1}V_0(\tilde{X}) = \frac{\partial}{\partial X^2}V_0(\tilde{X}) = 0$. For $X^1 \leq X^2 \leq z_1^1, V_1(\tilde{X}) = J_1(X^2, X^2)$, $\frac{\partial}{\partial X^2}V_1(\tilde{X}) \leq 0 = \frac{\partial}{\partial X^2}V_0(\tilde{X})$, $\frac{\partial}{\partial X^1}V_1(\tilde{X}) = \frac{\partial}{\partial X^1}V_0(\tilde{X}) = 0$ satisfying the basis for (3) and (4). Clearly, for $Y^1 \leq z_1^1$, $\frac{\partial}{\partial Y^1}\mathrm{E}V_1(\tilde{Y} - D) = 0$ and so $\frac{\partial}{\partial Y^1}J_2(\tilde{Y}) = \frac{\partial}{\partial Y^1}L(\tilde{Y}) = \frac{\partial}{\partial Y^1}J_1(\tilde{Y})$, satisfying the basis for (1). Finally, since $\frac{\partial}{\partial Y^2}\mathrm{E}V_1(\tilde{Y} - D) \leq 0$ for $Y^2 \leq z_1^1$, so $\frac{\partial}{\partial Y^2}J_2(\tilde{Y}) = h_2 + \frac{\partial}{\partial Y^2}\mathrm{E}V_1(\tilde{Y} - D) \leq h_2 = \frac{\partial}{\partial Y^2}J_1(\tilde{Y})$ for $Y^2 \leq z_1^1$. This proves the basis for (2).

For the induction step, assume (1)-(4) hold for $n-1$. By demonstrating each of the derivative dominance conditions, we are establishing that $z_{n-1}^1 \leq z_n^1$ and $z_{n-1}^2 \leq z_n^2$. We consider the cases where: I. $z_{n-1}^2 - z_{n-1}^1 < K_1$, and II. $z_{n-1}^2 - z_{n-1}^1 \geq K_1$.

I. We consider four main cases for examination: (A) $z_{n-1}^2 - K_1 \leq z_n^2 - K_1 \leq z_{n-1}^1$; (B) $z_{n-1}^1 \leq z_n^2 - K_1 \leq z_{n-1}^2$; (C) $z_{n-1}^2 \leq z_n^2 - K_1 \leq z_{n-1}^1 + K_1$; and (D) $z_{n-1}^1 + K_1 \leq z_n^2 - K_1$. Each case will also consider sub-cases, by starting inventory $\tilde{X}$.

The following logic holds for cases (A), (B), (C), and (D). For $X^1 \leq z_{n-1}^1$, $\frac{\partial}{\partial X^1}V_n(\tilde{X}) = 0 = \frac{\partial}{\partial X^1}V_{n-1}(\tilde{X})$ and for $z_{n-1}^1 \leq X^1 \leq z_n^1$, $\frac{\partial}{\partial X^1}V_n(\tilde{X}) = 0 \leq \frac{\partial}{\partial X^1}V_{n-1}(\tilde{X})$. Let us consider case (A) as representative of cases (A)-(D) and only consider it in depth. (The full proof, which details cases (A)-(D), may be obtained from the authors.)

When $z_n^1 \leq z_{n-1}^2$, X^2 can fall in the following intervals:

$$\begin{array}{ll}
X^2 < z_{n-1}^2 - K_1 & V_n(\tilde{X}) = J_n(X^2, X^2 + K_1), V_{n-1}(\tilde{X}) = J_{n-1}(X^2, X^2 + K_1) \\
& \dfrac{\partial}{\partial X^2}V_n(\tilde{X}) = \dfrac{\partial}{\partial Y^1}J_n(\tilde{Y})|_{Y^1 = X^2} + \dfrac{\partial}{\partial Y^2}J_n(\tilde{Y})|_{Y^2 = X^2 + K_1} \\
& \leq \dfrac{\partial}{\partial Y^1}J_{n-1}(\tilde{Y})|_{Y^1 = X^2} + \dfrac{\partial}{\partial Y^2}J_{n-1}(\tilde{Y})|_{Y^2 = X^2 + K_1} = \dfrac{\partial}{\partial X^2}V_{n-1}(\tilde{X}) \\
z_{n-1}^2 - K_1 \leq X^2 \leq z_n^2 - K_1 & V_n(\tilde{X}) = J_n(X^2, X^2 + K_1), V_{n-1}(\tilde{X}) = J_{n-1}(X^2, z_{n-1}^2) \\
& \dfrac{\partial}{\partial X^2}V_n(\tilde{X}) = \dfrac{\partial}{\partial Y^1}J_n(\tilde{Y})|_{Y^1 = X^2} + \dfrac{\partial}{\partial Y^2}J_n(\tilde{Y})|_{Y^2 = X^2 + K_1} \\
& \leq \dfrac{\partial}{\partial Y^1}J_{n-1}(\tilde{Y})|_{Y^1 = X^2} = \dfrac{\partial}{\partial X^2}V_{n-1}(\tilde{X}) \\
z_n^2 - K_1 \leq X^2 \leq z_{n-1}^1 & V_n(\tilde{X}) = J_n(X^2, z_n^2), V_{n-1}(\tilde{X}) = J_{n-1}(X^2, z_{n-1}^2) \\
& \dfrac{\partial}{\partial X^2}V_n(\tilde{X}) = \dfrac{\partial}{\partial Y^1}J_n(\tilde{Y})|_{Y^1 = X^2} \leq \dfrac{\partial}{\partial Y^1}J_{n-1}(\tilde{Y})|_{Y^1 = X^2} = \dfrac{\partial}{\partial X^2}V_{n-1}(\tilde{X})
\end{array}$$

$$
\begin{array}{ll}
z_{n-1}^1 \leq X^2 \leq z_n^1 & V_n(\tilde{X}) = J_n(X^2, z_n^2), V_{n-1}(\tilde{X}) = J_{n-1}(\max(z_{n-1}^1, X^1), z_{n-1}^2) \\
& \dfrac{\partial}{\partial X^2} V_n(\tilde{X}) = \dfrac{\partial}{\partial Y^1} J_n(\tilde{Y})|_{Y^1=X^2} \leq 0 = \dfrac{\partial}{\partial X^2} V_{n-1}(\tilde{X}) \\
z_n^1 \leq X^2 \leq z_{n-1}^2 & V_n(\tilde{X}) = J_n(z_n^1, z_n^2), V_{n-1}(\tilde{X}) = J_{n-1}(\max(z_{n-1}^1, X^1), z_{n-1}^2) \\
& \dfrac{\partial}{\partial X^2} V_n(\tilde{X}) = 0 = \dfrac{\partial}{\partial X^2} V_{n-1}(\tilde{X}) \\
z_{n-1}^2 \leq X^2 \leq z_n^2 & V_n(\tilde{X}) = J_n(z_n^1, z_n^2), V_{n-1}(\tilde{X}) = J_{n-1}(\max(z_{n-1}^1, X^1), X^2) \\
& \dfrac{\partial}{\partial X^2} V_n(\tilde{X}) = 0 \leq \dfrac{\partial}{\partial Y^2} J_{n-1}(\tilde{Y})|_{Y^2=X^2} = \dfrac{\partial}{\partial X^2} V_{n-1}(\tilde{X})
\end{array}
$$

When $z_n^1 > z_{n-1}^2$, the fifth range disappears, the range the fourth case covers is $z_{n-1}^1 \leq X^2 \leq z_{n-1}^2$, the range the sixth case covers is $z_n^1 \leq X^2 \leq z_n^2$, and another case appears:

$$
\begin{array}{ll}
z_{n-1}^2 \leq X^2 \leq z_n^1 & V_n(\tilde{X}) = J_n(X^2, z_n^2), V_{n-1}(\tilde{X}) = J_{n-1}(\max(z_{n-1}^1, X^1), z_{n-1}^2) \\
& \dfrac{\partial}{\partial X^2} V_n(\tilde{X}) = \dfrac{\partial}{\partial Y^2} J_n(\tilde{Y})|_{Y^2=X^2} \leq 0 \leq \dfrac{\partial}{\partial Y^2} J_{n-1}(\tilde{Y})|_{Y^2=X^2} = \dfrac{\partial}{\partial X^2} V_{n-1}(\tilde{X})
\end{array}
$$

II. $z_{n-1}^2 - z_{n-1}^1 \geq K_1$. The cases we consider are: (E) $z_{n-1}^1 \leq z_n^1 \leq z_{n-1}^2 - K_1$ and $z_{n-1}^1 \leq z_n^2 - K_1 \leq z_{n-1}^1 + K_1$; (F) $z_{n-1}^2 - K_1 \leq z_n^1 + K_1$ and $z_{n-1}^1 \leq z_n^2 - K_1 \leq z_{n-1}^1 + K_1$; (G) $z_{n-1}^1 + K_1 \leq z_n^1 \leq z_{n-1}^2$; (H) $z_{n-1}^1 + K_1 \leq z_n^2 - K_1 \leq z_{n-1}^2$ and $z_{n-1}^1 \leq z_n^1 \leq z_{n-1}^1 + K_1$; (I) $z_{n-1}^1 + K_1 \leq z_n^2 - K_1 \leq z_{n-1}^2$ and $z_{n-1}^2 \leq z_n^1$; and (J) $z_{n-1}^2 \leq z_n^2 - K_1$ and $z_n^2 - K_1 \leq z_n^1 \leq z_n^2$. Although the actual solution may be dependent upon the level of X^1, the derivative with respect to X^2 is identical in its form. Therefore, we present here the most representative case, (E).

For $X^1 \leq z_{n-1}^1$, $\frac{\partial}{\partial X^1} V_{n-1}(\tilde{X}) = 0$, for $z_{n-1}^1 \leq X^1 \leq z_{n-1}^2 - K_1$, $\frac{\partial}{\partial X^1} V_{n-1}(\tilde{X}) = \frac{\partial}{\partial Y^1} J_{n-1}(\tilde{Y})|_{Y^1=X^1} + \frac{\partial}{\partial Y^2} J_{n-1}(\tilde{Y})|_{Y^2=X^1+K_1} \geq 0$, and for $X^1 \geq z_{n-1}^2 - K_1$, $\frac{\partial}{\partial X^1} V_{n-1}(\tilde{X}) = \frac{\partial}{\partial Y^1} J_{n-1}(\tilde{Y})|_{Y^1=X^1} \geq 0$. Likewise, $X^1 \leq z_n^1$, $\frac{\partial}{\partial X^1} V_n(\tilde{X}) = 0$, and so $\frac{\partial}{\partial X^1} V_n(\tilde{X}) \leq \frac{\partial}{\partial X^1} V_{n-1}(\tilde{X})$ for $\tilde{X} \leq \tilde{z}_n$.

Now consider the subcases for X^2 in case (E).

$$
\begin{array}{ll}
X^2 \leq z_{n-1}^1 & V_n(\tilde{X}) = J_n(X^2, X^2 + K_1), V_{n-1}(\tilde{X}) = J_{n-1}(X^2, X^2 + K_1) \\
& \dfrac{\partial}{\partial X^2} V_n(\tilde{X}) = \dfrac{\partial}{\partial Y^1} J_n(\tilde{Y})|_{Y^1=X^2} + \dfrac{\partial}{\partial Y^2} J_n(\tilde{Y})|_{Y^2=X^2+K_1} \\
& \leq \dfrac{\partial}{\partial Y^1} J_{n-1}(\tilde{Y})|_{Y^1=X^2} + \dfrac{\partial}{\partial Y^2} J_{n-1}(\tilde{Y})|_{Y^2=X^2+K_1} = \dfrac{\partial}{\partial X^2} V_{n-1}(\tilde{X}) \\
z_{n-1}^1 \leq X^2 \leq z_n^1 & V_n(\tilde{X}) = J_n(X^2, X^2 + K_1), V_{n-1}(\tilde{X}) = J_{n-1}(z_{n-1}^1, z_{n-1}^1 + K_1) \\
& \dfrac{\partial}{\partial X^2} V_n(\tilde{X}) = \dfrac{\partial}{\partial Y^1} J_n(\tilde{Y})|_{Y^1=X^2} + \dfrac{\partial}{\partial Y^2} J_n(\tilde{Y})|_{Y^2=X^2+K_1} \\
& \leq 0 = \dfrac{\partial}{\partial X^2} V_{n-1}(\tilde{X})
\end{array}
$$

$$
\begin{array}{ll}
z_n^1 \le X^2 \le z_{n-1}^1 + K_1 & V_n(\tilde{X}) = J_n(z_n^1, z_n^1 + K_1), V_{n-1}(\tilde{X}) = J_{n-1}(z_{n-1}^1, z_{n-1}^1 + K_1), \\
& V_{n-1}(\tilde{X}) = J_{n-1}(X^1, X^1 + K_1) \\
& \frac{\partial}{\partial X^2} V_n(\tilde{X}) = 0 = \frac{\partial}{\partial X^2} V_{n-1}(\tilde{X}) \\
z_{n-1}^1 + K_1 \le X^2 \le z_n^1 + K_1 & V_n(\tilde{X}) = J_n(z_n^1, z_n^1 + K_1), V_{n-1}(\tilde{X}) = J_{n-1}(X^1, X^1 + K_1) \\
& \frac{\partial}{\partial X^2} V_n(\tilde{X}) = 0 = \frac{\partial}{\partial X^2} V_{n-1}(\tilde{X})
\end{array}
$$

Thus we have established (3) and (4). Consequently, we have the following for $\tilde{Y} \le \tilde{z}_n$:

$$
\begin{array}{rcl}
\frac{\partial}{\partial Y^1} V_n(\tilde{Y}) & \le & \frac{\partial}{\partial Y^1} V_{n-1}(\tilde{Y}) \\
\frac{\partial}{\partial Y^1} \mathrm{E} V_n(\tilde{Y} - D) & \le & \frac{\partial}{\partial Y^1} \mathrm{E} V_{n-1}(\tilde{Y} - D) \\
\frac{\partial}{\partial Y^1} \left[L(\tilde{Y}) + \beta \mathrm{E} V_n(\tilde{Y} - D) \right] & \le & \frac{\partial}{\partial Y^1} \left[L(\tilde{Y}) + \beta \mathrm{E} V_{n-1}(\tilde{Y} - D) \right] \\
\frac{\partial}{\partial Y^1} J_{n+1}(\tilde{Y}) & \le & \frac{\partial}{\partial Y^1} J_n(\tilde{Y}).
\end{array}
$$

The same logic applies for the derivatives with respect to Y^2, proving (1) and (2). This, thus implies z_n^1 and z_n^2 are monotonically non-decreasing in n.

(b) This part of the proof is achieved by comparing the derivatives of the value functions of the two systems. Since $V_0^A(\cdot) = V_0^B(\cdot) = 0$, $J_1^A(\tilde{Y}) = J_1^B(\tilde{Y}) = L(\tilde{Y})$ and thus all the derivatives are identical, resulting in $z_1^{1A} = z_1^{1B}$ and $z_1^{2A} = z_1^{2B}$. This establishes the basis. We now wish to prove the following:

$$
\frac{\partial}{\partial Y^1} J_n^B(\tilde{Y}) \le \frac{\partial}{\partial Y^1} J_n^A(\tilde{Y}) \text{ and } \frac{\partial}{\partial Y^2} J_n^B(\tilde{Y}) \le \frac{\partial}{\partial Y^2} J_n^A(\tilde{Y}).
$$

Assume these conditions for period $n-1$. There exist four possible cases in period $n-1$: (i) $z_{n-1}^{2A} - \zeta_{n-1}^A \le K_1^A, z_{n-1}^{2B} - \zeta_{n-1}^B \le K_1^B$; (ii) $z_{n-1}^{2A} - \zeta_{n-1}^A > K_1^A, z_{n-1}^{2B} - \zeta_{n-1}^B \le K_1^B$; (iii) $z_{n-1}^{2A} - \zeta_{n-1}^A \le K_1^A, z_{n-1}^{2B} - \zeta_{n-1}^B > K_1^B$; and (iv) $z_{n-1}^{2A} - \zeta_{n-1}^A > K_1^A, z_{n-1}^{2B} - \zeta_{n-1}^B > K_1^B$. Lemma A1 ensures that $z_{n-1}^{1A} \le z_{n-1}^{1B}$ in all these cases. While these four cases are exhaustive, they are very similar in their logic and for the sake of brevity only case (i) will be examined. The full proof may be obtained from the authors.

A single representative example from case (i) will be demonstrated; all other situations mimic the following subcases closely in principle.

We deal with an example where $z_{n-1}^{1B} \le z_{n-1}^{2A}$. We consider various sub-cases which exhaust all potential starting inventory positions. These inventory positions are for $\tilde{X} \in S^B := \{\tilde{Y} \in$

$\Re^2 | Y^2 - Y^1 \leq K_1^B\}$. Firstly, we consider the derivatives with respect to X^1

$$\begin{array}{rl}
X^1 \leq z_{n-1}^{1A} & \frac{\partial}{\partial X^1} V_{n-1}^A(\tilde{X}) = 0 = \frac{\partial}{\partial X^1} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{1A} \leq X^1 \leq z_{n-1}^{1B} & \frac{\partial}{\partial X^1} V_{n-1}^A(\tilde{X}) = \frac{\partial}{\partial Y^1} J_{n-1}^A(\tilde{Y})|_{Y^1=X^1} \geq 0 = \frac{\partial}{\partial X^1} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{1B} \leq X^1 & \frac{\partial}{\partial X^1} V_{n-1}^A(\tilde{X}) = \frac{\partial}{\partial Y^1} J_{n-1}^A(\tilde{Y})|_{Y^1=X^1} \geq \frac{\partial}{\partial Y^1} J_{n-1}^B(\tilde{Y})|_{Y^1=X^1} = \frac{\partial}{\partial X^1} V_{n-1}^B(\tilde{X})
\end{array}$$

and those with respect to X^2

$$\begin{array}{rl}
X^2 \leq z_{n-1}^{2A} - K_1^A & \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}) = \frac{\partial}{\partial Y^1} J_{n-1}^A(\tilde{Y})|_{Y^1=X^2} + \frac{\partial}{\partial Y^2} J_{n-1}^A(\tilde{Y})|_{Y^2=X^2+K_1^A} \\
& \geq \frac{\partial}{\partial Y^1} J_{n-1}^A(\tilde{Y})|_{Y^1=X^2} + \frac{\partial}{\partial Y^2} J_{n-1}^A(\tilde{Y})|_{Y^2=X^2+K_1^B} \geq \\
& \frac{\partial}{\partial Y^1} J_{n-1}^B(\tilde{Y})|_{Y^1=X^2} + \frac{\partial}{\partial Y^2} J_{n-1}^B(\tilde{Y})|_{Y^2=X^2+K_1^B} = \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{2A} - K_1^A \leq X^2 \leq z_{n-1}^{1A} & \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}) = \frac{\partial}{\partial Y^1} J_{n-1}^A(\tilde{Y})|_{Y^1=X^2} \geq \\
& \frac{\partial}{\partial Y^1} J_{n-1}^B(\tilde{Y})|_{Y^1=X^2} + \frac{\partial}{\partial Y^2} J_{n-1}^B(\tilde{Y})|_{Y^2=X^2+K_1^B} = \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{1A} \leq X^2 \leq z_{n-1}^{2B} - K_1^B & \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}) = 0 \geq \\
& \frac{\partial}{\partial Y^1} J_{n-1}^B(\tilde{Y})|_{Y^1=X^2} + \frac{\partial}{\partial Y^2} J_{n-1}^B(\tilde{Y})|_{Y^2=X^2+K_1^B} = \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{2B} - K_1^B \leq X^2 \leq z_{n-1}^{1B} & \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}) = 0 \geq \frac{\partial}{\partial Y^1} J_{n-1}^B(\tilde{Y})|_{Y^1=X^2} = \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{1B} \leq X^2 \leq z_{n-1}^{2A} & \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}) = 0 = \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{2A} \leq X^2 \leq z_{n-1}^{2B} & \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}) = \frac{\partial}{\partial Y^2} J_{n-1}^A(\tilde{Y})|_{Y^2=X^2} \geq 0 = \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X}) \\
z_{n-1}^{2B} \leq X^2 & \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}) = \frac{\partial}{\partial Y^2} J_{n-1}^A(\tilde{Y})|_{Y^2=X^2} \\
& \geq \frac{\partial}{\partial Y^2} J_{n-1}^B(\tilde{Y})|_{Y^2=X^2} = \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X})
\end{array}$$

With the many subcases of (i)-(iv), this suffices to show

$$\frac{\partial}{\partial X^1} V_{n-1}^B(\tilde{X}) \leq \frac{\partial}{\partial X^1} V_{n-1}^A(\tilde{X}) \text{ and } \frac{\partial}{\partial X^2} V_{n-1}^B(\tilde{X}) \leq \frac{\partial}{\partial X^2} V_{n-1}^A(\tilde{X}).$$

Follow the logic for the derivatives with respect to Y^1 (the same logic holds for Y^2):

$$\begin{array}{rcl}
\frac{\partial}{\partial Y^1} V_{n-1}^B(\tilde{Y}) & \leq & \frac{\partial}{\partial Y^1} V_{n-1}^A(\tilde{Y}) \\
\frac{\partial}{\partial Y^1} \mathrm{E} V_{n-1}^B(\tilde{Y} - D) & \leq & \frac{\partial}{\partial Y^1} \mathrm{E} V_{n-1}^A(\tilde{Y} - D) \\
\frac{\partial}{\partial Y^1} \left[L(\tilde{Y}) + \beta \mathrm{E} V_{n-1}^B(\tilde{Y} - D) \right] & \leq & \frac{\partial}{\partial Y^1} \left[L(\tilde{Y}) + \beta \mathrm{E} V_{n-1}^A(\tilde{Y} - D) \right]
\end{array}$$

$$\frac{\partial}{\partial Y^1} J_n^B(\tilde{Y}) \quad \leq \quad \frac{\partial}{\partial Y^1} J_n^A(\tilde{Y}).$$

Using Lemma A1, this implies the result. ■

Lemma A2 *Let $N = 2$ supply chain with $K_2 \geq K_1$ operate under a base-stock policy with levels $\tilde{z}$ such that $z^2 - z^1 = K_1$ and let $\tilde{X} \leq z$. If the lower echelon base-stock level is reached, then the upper echelon can also achieve its base-stock level in the same period.*

Proof If z^1 is reached, $z^1 - X^1 \leq X^2 - X^1$ and $z^1 - x^1 \leq K_1$. Since $z^2 - z^1 = K_1$, $z^2 - K_1 \leq X^2$, or $0 \leq z^2 - X^2 \leq K_1 \leq K_2$, and thus z^2 can be reached in the current period. ■

Lemma A3 *Consider an N-stage serial supply chain operating under a MEBS policy where $z^j - z^{j-1} = K_1$ and $K_j \geq K_1$ for all $j > 1$. Assume for a given period $X^j - X^{j-1} = K_1$ for all $j > 1$. Then (a) $X^j - X^{j-1} = K_1$ for all $j > 1$ in all future periods; and (b) once $z^1 - K_1 \leq X^1 \leq z^1$, then $\tilde{z}$ is reachable in a single period for all echelons.*

Proof (a) Subscripts in this proof will reflect time counted backwards (but does not reflect a finite-time horizon). In any period n for installation $1 < j < N$, we have

$$\begin{aligned}
X_{n-1}^j - X_{n-1}^{j-1} &= Y_n^{j*} - Y_n^{(j-1)*} = \min(z^j, Y_n^{(j-1)*} + K_1, X_n^{j+1}) - Y_n^{(j-1)*} \\
&= \min(z^j, Y_n^{(j-1)*} + K_1, X_n^j + K_1) - Y_n^{(j-1)*} \\
&= Y_n^{(j-1)*} + K_1 - Y_n^{(j-1)*} = K_1
\end{aligned}$$

The fourth equality is drawn from two facts: (a) $Y_n^{(j-1)*} \leq z^{j-1}$ and hence $Y_n^{(j-1)*} + K_1 \leq z^{j-1} + K_1 = z^j$; and (b) $Y_n^{(j-1)*} \leq X_n^j$ and hence $Y_n^{(j-1)*} + K_1 \leq X_n^j + K_1 = X_n^{j+1}$.

Similarly, for echelon N using fact (a) we have,

$$\begin{aligned}
X_{n-1}^N - X_{n-1}^{N-1} &= Y_n^{N*} - Y_n^{(N-1)*} = \min(z^N, Y_n^{(N-1)*} + K_1) - Y_n^{(N-1)*} \\
&= Y_n^{(N-1)*} + K_1 - Y_n^{(N-1)*} = K_1
\end{aligned}$$

(b) Whether the system can reach $\tilde{z}$ in a single period is simply an issue of stock availability in the system. If $X^1 \leq z^1 \leq X^1 + K_1$ and $X^2 - X^1 = K_1$, then clearly $X^2 \geq z^1$ and so echelon 1 can reach z^1 in the current period. Likewise, each of the higher installations, j, can replenish up to its echelon level z^j. ■

Lemma A4 *Consider an N-stage serial multi-echelon system with capacity limits of $K_i \geq K_1$ at stage i and $0 < E[D] < K_1$. Consider the echelon base-stock policy, $\pi(K)$, ordering up to $(i-1)K_1$ at echelon i. Define T as the random variable representing the number of periods between subsequent visits to these base-stock levels. Then (a) $E[T] < \infty$ and (b) there exists $A \in \Re_+$ such that $J_n(0, K_1, 2K_1, \ldots, (N-1)K_1) \leq An$.*

Proof Lemma A3 shows that once the level of K_1 units is achieved at each installation higher than installation 1, these levels will be retained in all future periods. Since $K_1 \leq K_i$ for all $i \in \{2, \ldots, N\}$, there will not be a constraining capacity other than K_1 under policy $\pi(K)$. Therefore, after the installation inventory level of K_1 is reached at all installations above installation 1, the system can be analyzed identically to that of Kapuscinski and Tayur (1995). Installation 1 can be regarded as a capacitated single installation since there will never be a shortage of supply from installation 2 ($z^2 - z^1 = K_1$). Consequently, using part (a) of Lemma A in Kapuscinski and Tayur (1995) under a stationary context, $\mathrm{E}[T] < \infty$.
(b)

$$\begin{aligned}
J_n(0, K_1, 2K_1, \ldots, (N-1)K_1) &\leq J_n^{\pi(K),\beta=1}(0, K_1, 2K_1, \ldots, (N-1)K_1) \\
&= \mathrm{E}\left[\sum_{i=0}^{n}\left[p|y_i^1 - D_i| + \sum_{j=2}^{N} h_j(y_i^j - y_i^{j-1})\right]\right] \\
&\leq \mathrm{E}\left[\sum_{i=0}^{n}\left[p|y_i^1| + p|D_i| + \sum_{j=2}^{N} h_j K_1\right]\right] \\
&= \sum_{i=0}^{n}\left[p\mathrm{E}|y_i^1| + p\mathrm{E}[D_i] + \sum_{j=2}^{N} h_j K_1\right] \\
&\leq \sum_{i=0}^{n}\left[p\mathrm{E}[T]K_1 + pK_1 + \sum_{j=2}^{N} h_j K_1\right] \\
&= \left(p\mathrm{E}[T] + p + \sum_{j=2}^{N} h_j\right) K_1 n = An
\end{aligned}$$

where A is a finite real number since $\mathrm{E}[T] < \infty$ from (a). ■

Theorem 10 *There exists a stationary policy δ^* which is average optimal in the sense that $\Phi(\delta^*, x) := \limsup_{n\to\infty} \frac{1}{n} E_{\tilde{X},\delta^*}[\sum_{m=0}^{n-1} L(\tilde{X}_m)] = \inf_{\tilde{X}\in S} \inf_\delta \Phi(\delta, \tilde{X}) =: g$ for all $\tilde{X}$. δ^* is limit* *discount optimal in that for any* $\tilde{X} \in S$ *for all* $\beta \nearrow 1$ *such that* $\delta^*(\tilde{X}) = \lim_{\beta \nearrow 1} \delta^*_\beta(\tilde{X})$. *Additionally,* $g = \lim_{\beta \nearrow 1}(1-\beta)m_\beta = \lim_{\beta \nearrow 1}(1-\beta)V_\beta$.

Proof We demonstrate that the conditions described by Schäl (1993) are satisfied permitting us to invoke the main result from that paper. The techniques used here could also be applied to the Federgruen and Zipkin (1986a) model, resulting in a simpler proof.

The finiteness of $\mathcal{A}(\tilde{X})$ follows from the integrality of the state space and the capacity limits at each installation. This satisfies the first of Schäl's conditions. We now demonstrate the second condition, (B). We use Lemma **??** to bound $w_\beta(\tilde{X})$ by constructing a policy δ for which right-hand side is finite.

Let $\varphi_\beta(\tilde{X})$ be the randomized stationary policy such that

$$\underline{V}_\beta(\tilde{X}) = \min_{\tilde{Y} \in \mathcal{A}(\tilde{X})} \mathrm{E}V_\beta(\tilde{Y} - D) = \mathrm{E}V_\beta(\varphi_\beta(\tilde{X}) - D).$$

exists. Let us define a random variable $\sigma^\beta := \inf\{n \geq 0, \beta\underline{V}_\beta(\tilde{X}_n) \leq m_\beta\}$. We show that there exist a finite upper bound on σ^β, which is independent of β.

The optimal base-stock values of V_β are uniformly bounded for $0 \leq \beta < 1$ (see Theorem **??**); let us denote this bound as $\tilde{z} < \infty$. Define the set $\mathcal{H} := \{\tilde{X} \in S | 0 \leq \tilde{X} \leq \tilde{z}\}$. Let $H = |\mathcal{H}|$, the cardinality of $\mathcal{H}$. Clearly, H is finite since $\tilde{z} < \infty$ and the state space consists of integers. Since V_β is continuous, convex, and increases to ∞ when any coordinate of its argument goes to +/-∞, there exists a finite minimizer $\tilde{Y}_\beta \in \mathcal{H}$ such that

$$\underline{m}_\beta = \inf_{\tilde{Y}} \mathrm{E}[V_\beta(\tilde{Y} - D)] = \mathrm{E}[V_\beta(\tilde{Y}_\beta - D)].$$

It is easy to show that $\beta\underline{V}_\beta \leq V_\beta$ and $\beta\underline{m}_\beta \leq m_\beta \leq \underline{m}_\beta$ (see Schäl, 1993, for a formal derivation). Therefore,

$$\begin{aligned} \sigma^\beta &= \inf\{n \geq 0, \beta\underline{V}_\beta(\tilde{X}_n) \leq m_\beta\} \\ &\leq \inf\{n \geq 0, \beta\underline{V}_\beta(\tilde{X}_n) \leq \beta\underline{m}_\beta\} \\ &= \inf\{n \geq 0, \inf_{\tilde{Y} \in \mathcal{A}(\tilde{X}_n)} \mathrm{E}[V_\beta(\tilde{Y} - D)] = \underline{m}_\beta\}. \end{aligned}$$

To generate an upper bound on σ^β, consider a non-stationary policy, δ, where each element of $\mathcal{H}$ will sequentially assume the base-stock role as the prior one is reached. Since $\tilde{Y}_\beta \in \mathcal{H}$, we show that this non-stationary policy is guaranteed to visit $\tilde{Y}_\beta$. Glasserman and Tayur (1994, p.917) demonstrate that shortfall states are Harris-ergodic. Since E$D < K_1$, for a given (fixed) base-stock level, the process we consider is Harris-ergodic for the subset of all non-transient states. Consequently, we have finite mean recurrence times. Resulting from this, if a state can be reached, then the expected time to get from any state within $\mathcal{H}$ to another state in $\mathcal{H}$ is finite. Considering a sequence of base-stock levels corresponding to the inventory target, the expected time to visit that inventory target is finite. Let $T(\tilde{X})$ be the expected time to visit all points in $\mathcal{H}$. Let $\tau_0(\tilde{X}) = \inf\{n \geq 0, \sum_{i=1}^{n} D_i \geq \tilde{X}\}$ and $\tau_i(i-1)$ be the first visit time to visit point $i \in \{1, 2, \ldots, H\}$ when starting from position $i-1$. As stated,

$$\sigma^{\beta} \leq \tau_0(\tilde{X}) + \sum_{i=1}^{H} \tau_i(i-1) =: T(\tilde{X})$$

where $\tau_1(0)$ begins from a position below 0, having traversed there from the initial stock position $\tilde{X}$ in $\tau_0(\tilde{X})$ periods. Since H is finite, $\sigma = T(\tilde{X})$, which is greater than or equal to σ^{β}, is finite. Policy δ is the non-stationary policy where each element of $\mathcal{H}$ sequentially becomes a base-stock level and continues to be base stock until it is visited.

Now, $\mathrm{E}L(\varphi(\tilde{X}_\sigma)) \leq \mathrm{E}[(p,h_1)^+|Y_\sigma^1 - D| + \sum_{j=2}^{N} h_j K_1] \leq E[(p,h_1)^+|Y_\sigma^1| + (p,h_1)^+|D| + \sum_{j=2}^{N} h_j K_1] \leq (p,h_1)^+ z^1 + (p,h_1)^+ K_1 + \sum_{j=2}^{N} h_j K_1 < \infty$ since Y_σ^1 is guaranteed to be within $\mathcal{H}$. Hence,

$$\begin{aligned}
&\mathrm{E}_{\tilde{X},\delta}\left[\sum_{n=0}^{\sigma-1} L(\tilde{X}_n) + L(\varphi(\tilde{X}_\sigma))\right] \\
&\leq \mathrm{E}_{\tilde{X},\delta}\left[\sum_{n=0}^{T(\tilde{X})-1} [(p,h_1)^+|Y_n^1 - D_n| + \sum_{j=2}^{N} h_j K_1] + L(\varphi(\tilde{X}_\sigma))\right] \\
&\leq \mathrm{E}_{\tilde{X},\delta}\left[\sum_{n=0}^{T(\tilde{X})-1} [(p,h_1)^+|Y_n^1| + (p,h_1)^+|D_n| + \sum_{j=2}^{N} h_j K_1] + L(\varphi(\tilde{X}_\sigma))\right] \\
&\leq \mathrm{E}_{\tilde{X},\delta}\left[\sum_{n=0}^{T(\tilde{X})-1} (p,h_1)^+|Y_n^1| + (p,h_1)^+ K_1 + \sum_{j=2}^{N} h_j K_1\right] + \mathrm{E}[L(\varphi(\tilde{X}_\sigma))] \\
&= \mathrm{E}_{\tilde{X},\delta}\left[\sum_{n=0}^{T(\tilde{X})-1} (p,h_1)^+|Y_n^1|\right] + [(p,h_1)^+ K_1 + \sum_{j=2}^{N} h_j K_1]\mathrm{E}[T(\tilde{X})] + \mathrm{E}[L(\varphi(\tilde{X}_\sigma))] \qquad (5)
\end{aligned}$$

by Wald's Theorem. The first inequality comes from the existence of $\tilde{Y}_\beta$ in $\mathcal{H}$. The second inequality results from the triangle inequality. The third inequality follows from the fact that $\mathrm{E}[D] < K_1$. Glasserman and Tayur (1994) establish that the shortfalls, and hence the inventory levels themselves, are stationary in distribution and finite, almost surely, for an ergodic stationary demand with $\mathrm{E}[D] < K_1$.

For inventory x, shortfall s, and up-to level y: (a) Aviv and Federgruen (1997) show $\mathrm{E}[|x|^\rho] < \infty$ and $\mathrm{E}[|s|^\rho] < \infty$ when $\mathrm{E}[D^l] < \infty$ for all $l \leq \rho + 1$; (b) Kapuscinski and Tayur (1998) show $\mathrm{E}[|x|^\rho] < \infty$ and $\mathrm{E}[|y|^\rho] < \infty$ when $\mathrm{E}[D^{2\rho+2}] < \infty$; and (c) Simchi-Levi and Zhao (2001) show $\mathrm{E}[|x|^\rho] < \infty$, $\mathrm{E}[|y|^\rho] < \infty$, and $\mathrm{E}[|s|^\rho] < \infty$ when $\mathrm{E}[D^l] < \infty$ for all $l \leq 2\rho$. For $\rho = 1$, (a) and (c) are sufficient to show that the expected cost in one period is finite for $0 < \mathrm{E}[D] < K_1$ and $\mathrm{E}[D^2] < \infty$. This combined with the positive recurrence of the shortfall Markov chain (Glasserman and Tayur, 1994) establishes the finiteness of the first term in (5). This also demonstrates the finiteness of the second term. Consequently, (5) is finite since $\mathrm{E}[L(\varphi(\tilde{X}_\sigma))] < \infty$ from the logic above. Therefore, by Lemma **??**, we have w_β bounded for all $\beta < 1$, thus satisfying (B). ■