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Online Appendices for

Managing a Non-Cooperative Supply Chain with Limited Capacity

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Theorem 2 *Assume $p_2 > h_2(1-\beta)/\beta$. Let salvage value functions $S_0^1(\tilde{X}) = \lambda_1(X^1 - \gamma_1)^2$ and $S_1^2(\tilde{X}) = \lambda_2(\gamma_2 - X^2)^+$. There exist $\lambda_1, \lambda_2, \gamma_1$, and γ_2 such that for each starting inventory $\tilde{X} \in \mathcal{B}_k$:*

(i) using Pareto refinement, there exists a unique pure-strategy Nash equilibrium, which is a modified echelon base-stock policy, in period k;

(ii) $V_k^j(\tilde{X}) = V_k^{j1}(X^1) + V_k^{j2}(X^2)$ for $j = 1, 2$; and

(iii) $z_{k+1}^j \leq z_k^j$ for $(j = 1, k \geq 1)$ and $(j = 2, k \geq 2)$.

Proof This proof is by induction. In addition to showing that the main conditions (i)-(iii) hold, we also use the following conditions as a part of the inductional loop for $\tilde{Y} \in \mathcal{B}_k$: (iv) J_{k+1}^1 is convex in Y^1 for $Y^1 \leq z_k^1$; (v) J_{k+1}^1 is convex decreasing in $Y^2 \leq z_k^2$, and decreasing in $Y^2 \geq z_k^2$; (vi) J_{k+1}^2 is convex in Y^2; (vii) J_{k+1}^2 is convex decreasing in $Y^1 \leq (\min(z_k^2 - K_1, z_k^1), \underline{z}_k^1)^+$, and decreasing in $Y^1 \geq (\min(z_k^2 - K_1, z_k^1), \underline{z}_k^1)^+$; (viii) $J_{k+1}^i(Y^1, Y^2) = J_{k+1}^{i1}(Y^1) + J_{k+1}^{i2}(Y^2)$ for $i \in \{1, 2\}$ and $\underline{z}_{k+1}^1 := \arg\min_{Y^1} J_{k+1}^{11}(Y^1)$; (ix) $\underline{z}_{k+1}^1 \leq z_{k+1}^1$; (x) $\partial_j J_{k+1}^i(\tilde{Y}) \geq \partial_j J_k^i(\tilde{Y})$ for $Y^j \leq z_k^j$ when $(i, j) \in \{(1, 1), (1, 2), (2, 2)\}$ and for $Y^1 \leq \min(z_k^2 - K_1, z_k^1)$ when $(i, j) = (2, 1)$; (xi) $\underline{z}_{k+1}^1 = y_{my}^*$; and outside the band (xii) for $Y^2 \geq Y^1 + K_1$, J_{k+1}^1 is convex in Y^1, J_{k+1}^2 is increasing in Y^2, $\partial_1 J_{k+1}^1 \leq 0$ for $\{Y^1 + K_1 \leq Y^2 \leq z_{k+1}^1 + 2K_1, Y^1 \leq z_{k+1}^1\}$, $\partial_2 J_{k+1}^1 = 0$ for $Y^1 + K_1 \leq Y^2 \leq z_k^2$, and for $\tilde{Y} \in \mathcal{B}^{K_1}$, we have $J_{k+1}^1(\tilde{Y}) = J_{k+1}^{11}(Y^1) + J_{k+1}^{12}(Y^2)$;[1] and outside the band (xiii) for $Y^2 \geq Y^1 + K_1$, $r_{k+1}^1(Y^2) = z_{k+1}^1$ for $Y^2 \in [z_{k+1}^1 + K_1, z_{k+1}^1 + 2K_1]$ and $r_{k+1}^1(Y^2) \leq Y^2 - 2K_1$ for $Y^2 > z_{k+1}^1 + 2K_1$.

Some of these conditions correspond to the conditions in the statement of Lemma 2: inductional statements (iv) and (vi) correspond to Lemma 2's condition (a); statement (viii) corresponds to condition (b); and inductional statement (xii) corresponds to condition (d). Conditions (v), (vii), and (xii) warrant special comment. The firms' cost functions are decreasing (weakly) in the other firm's inventory level since, when the retailer holds more inventory, she reduces the supplier's cost of the consumer backlog; when the supplier holds more inventory, he reduces the chance of the retailer being starved of material. Both of these circumstances result in reduced costs for the other firm. (xiii) means that firm 1's best reply function rises vertically along the X^2 dimension for a height of K_1 above the band $\mathcal{B}$ and for greater values of Y^2 it will never return to the band, thus eliminating the possibility of additional equilibria at higher supplier stocking levels.

We start with the inductional step to clearly present the logic of the critical elements of the proof. In order for the inductional step to hold, one of the elements is condition (iii) that the "up-to" values, z_n^js, are decreasing in the number of remaining periods n. The initial steps for $n = 1, 2$ will make it possible - we present them after the inductional step, since they use mostly the same logic.

<u>*Induction step, Period n:*</u> Assume (i) -(xiii) above with index $n-1$ replacing k. Note from (viii), J_n^2 separates $(J_n^2(\tilde{Y}) = J_n^{21}(Y^1) + J_n^{22}(Y^2))$ which immediately delivers a proper definition[2] of $z_n^2 = \arg\min_{Y^2} J_n^{22}(Y^2)$, due to the convexity of J_n^{22}.

[1] J^{11} and J^{12} do not have the same values in the band $\mathcal{B}$ and in $\mathcal{B}^{K_1}$.

[2] In Definition 3, z_n^2 was defined as not dependent on Y^1.

• Unconstrained response functions:
Each of the three boxes in Figure 2 illustrates the feasible set, $\mathcal{A}(\tilde{X}) = \times_{j=1}^{2}\mathcal{A}^j(\tilde{X})$. Note that echelon 2's controllable costs are minimized at z_n^2 and that echelon 2 has no incentive to store more than K_1; that is, the supplier's best response function remains in the band. Consequently, from Lemma 2, the unconstrained best-reply function (defined as in Lemma 2, that is ignoring the current period initial constraints while accounting for the discounted future expected costs) is

$$r_n^2 = z_n^2|[Y^1, Y^1 + K_1].$$

The retailer may be better off choosing actions outside of band $\mathcal{B}$ and consequently, her response function may depart from it. *If* the response function is limited to the band,

$$r_n^1 = \underline{z}_n^1|[Y^2 - K_1, Y^2].$$

From induction assumption (xiii), $r_n^1 = z_n^1$ for $z_n^1 + K_1 \leq Y^2 \leq z_n^1 + 2K_1$ and, from (xii), r_n^1 is within the band for $Y^2 \leq z_n^1 + K_1$. Thus, from Lemma 2, $r_n^1 = \min\{\underline{z}_n^1|[Y^2 - K_1, Y^2], z_n^1\}$ for $Y^2 \leq z_n^1 + K_1$. Also, note that $\underline{z}_n^1 = y_{my}^*$ from induction assumption (xi).
• Unconstrained equilibria:
We consider first the unconstrained response functions and unconstrained equilibrium, which ignore the capacity constraints. The conditions of Theorem 1 (strategy spaces are nonempty compact convex subsets of Euclidean space, payoff functions are continuous and quasi-convex in Y^i) are satisfied for a given starting inventory position, $\tilde{X} \in \mathcal{B}$; there exists at least one pure strategy equilibrium. As illustrated in Figure 1(C), a unique equilibrium exists, if $y_{my}^* = \underline{z}_n^1 \leq z_n^2 \leq \underline{z}_n^1 + K$, for $n \geq 2$. Otherwise, the unconstrained response functions overlap over a range $[Y^1, Y^1 + K_1]$ for $y_{my}^* \leq Y^1 \leq \min(z_n^1, z_n^2 - K_1)$, resulting in multiple equilibria. Recall that Pareto refinement discards all equilibria with higher costs for both players. Here, the use of Pareto refinement results in a single undominated equilibrium: Clearly conditions (v) and (vii) apply for the multiple-equilibria interval and, thus, J_n^i is non-increasing in Y^{-i} (Y^{-i} refers to the player who is not i), implying that there exists an equilibrium with lower costs for each of the players, which is $(Y^1, Y^1 + K_1)$, where $Y^1 = \min(z_n^1, z_n^2 - K_1)$.
• Constrained equilibria:
Given the unconstrained equilibria, the properties (monotonicities) of individual value functions, J_n^i, and of response functions, we now describe the constrained equilibria (actual equilibria given the initial state). We use the three possible cases, shown in Figure 1, in which these best-reply functions can interact: (A) $z_n^1 \leq z_n^2 - K_1$, (B) $\underline{z}_n^1 < z_n^2 - K_1 \leq z_n^1$, and (C) $z_n^2 - K_1 \leq \underline{z}_n^1 \leq z_n^2 \leq \underline{z}_n^1 + K_1$. In the case when $z_n^2 < \underline{z}_n^1$, the dynamics are very similar to the extreme case of $z_n^2 = \underline{z}_n^1$ within case (C). Since the logic is the same, we do not explicitly consider this case here. The restrictions on Y^1 in induction statement (vii) are intended to guarantee that all the properties are valid for $Y^1 \leq \min(z_n^2 - K_1, z_n^1)$, relating to Figure 1(A)(B). However, in case (C), $\underline{z}_n^1$ defines the equilibrium. Therefore, all properties are proved in the area up to the equilibrium. Considering case (A) first, we find the equilibrium: $(z_n^1|[X^1, X^2], z_n^1|[X^1, X^2] + K_1)$. Likewise for case (B): $((z_n^2 - K_1)|[X^1, X^2], (z_n^2 - K_1)|[X^1, X^2] + K_1)$. For case (C): $(\underline{z}_n^1|[X^1, X^2], z_n^2|[X^2, X^2 + K_1])$. We now show (i), (ii), (iv), (v), (vi), (vii):
The Pareto dominating equilibrium is clearly a modified echelon base-stock policy, $Y_n^1 = \min(z_n^1, z_n^2 - K_1)|[X_n^1, X_n^2]$ for cases (A) and (B), $Y_n^1 = \underline{z}_n^1|[X_n^1, X_n^2]$ for case (C), and $Y_n^2 = z_n^2|[X_n^2, Y_n^1 + K_1]$, demonstrating (i). Due to assumed convexities and separabilities, now we show that V_n^i is separable, J_{n+1}^i is convex in Y^i, and J_{n+1}^1 is convex in $Y^2 \leq z_n^2$ and J_{n+1}^2 is convex for $Y^1 \leq \min(z_n^2 - K_1, z_n^1)$:
Consider case (A). Since J_n^1 is convex in Y^1 and convex decreasing in $Y^2 \leq z_n^2$, the function $J_n^1(Y^1, Y^1 + K_1)$ is convex for $Y^1 \leq z_n^2 - K_1$, minimized at $Y^1 = z_n^1$, where $z_n^1 \leq z_n^2 - K_1$. Point

(ii) for case (A) follows, as the equilibrium value function depends only on Y^1, for $X^2 \leq z_n^2 - K_1$. Due to the feasibility constraints, $Y^1 = X^2$ below the equilibrium up-to levels. In order to show properties of J_{n+1}^i, it is sufficient to show them for functions V_n^i: L^i is convex and separable (not influencing J_{n+1}^{3-i}) and the demand operator shifts $\beta \mathrm{E}[V_n^i(\tilde{Y} - \tilde{D})]$ downwards, so adding these two components results in all desired properties for J_{n+1}^i. V_n^1 is flat (constant valued) in $X^1 \leq z_n^1$ and convex increasing in $z_n^1 \leq X^1$ due to the following.

Note that $\partial_1 V_n^1(\tilde{X}) = \mathcal{D}J_n^{11}(Y^1)|_{Y^1=X^1} + \mathcal{D}J_n^{12}(Y^2)|_{Y^2=X^1+K_1}$ for $z_n^1 \leq X^1 \leq z_n^2 - K_1$ and $\partial_1 V_n^1(\tilde{X}) = \mathcal{D}J_n^{11}(Y^1)|_{Y^1=X^1}$ for $z_n^2 - K_1 \leq X^1$. Since $\mathcal{D}J_n^{12}(Y^2)|_{Y^2=X^1+K_1} \leq 0$ for $z_n^1 \leq X^1 \leq z_n^2 - K_1$ and both terms are non-decreasing, convexity follows thus implying (iv). V_n^1 is convex decreasing in $X^2 \leq z_n^1$ and flat in $z_n^1 \leq X^2 \leq z_n^2$, and V_n^1 is decreasing in $X^2 \geq z_n^2$ from inductional assumption (v) for $k = n-1$, implying (v) for $k = n$. V_n^2 is convex decreasing in $X^2 \leq z_n^1$ and flat in $z_n^1 \leq X^2 \leq z_n^2$, and convex increasing in $X^2 \geq z_n^2$. V_n^2 is flat in $X^1 \leq z_n^1$, decreasing in $z_n^1 \leq X^1$, implying (vi) and (vii). The logic is similar for case (B). V_n^1 is flat in $X^1 \leq z_n^2 - K_1$ and convex increasing in $X^1 \geq z_n^2 - K_1$ (implying (iv)). V_n^1 is convex decreasing in $X^2 \leq z_n^2 - K_1$, flat in $z_n^2 - K_1 \leq X^2 \leq z_n^2$, and decreasing in $X^2 \geq z_n^2$ (implying (v)). V_n^2 is convex decreasing in $X^2 \leq z_n^2 - K_1$, flat in $z_n^2 - K_1 \leq X^2 \leq z_n^2$, and convex increasing in $X^2 \geq z_n^2$ (justifying (vi)). V_n^2 is flat in $X^1 \leq z_n^2 - K_1$ and decreasing in $X^1 \geq z_n^2 - K_1$ ((vii)). Similarly, for case (C) V_n^1 is flat for $X^1 \leq \underline{z}_n^1$, convex increasing for $X^1 > \underline{z}_n^1$ ((iv)), convex decreasing for $X^2 \leq z_n^2 - K_1$, and decreasing in $X^2 > z_n^2 - K_1$ ((v)). Also, V_n^2 is flat for $X^1 \leq \underline{z}_n^1$, decreasing for $X^1 > \underline{z}_n^1$, convex decreasing for $X^2 \leq z_n^2 - K_1$ ((vii)), and convex increasing in $X^2 > z_n^2 - K_1$ ((vi)). Note that J_n^i is separable from induction assumption (ii) and the separability of L^i. The separability of V_n^i follows due to the fact that the equilibrium in period n depends upon X^1 or X^2 but not a combination of both within a single parameter. This shows (ii).

• Ordering of derivatives and up-to levels:

Let us define the following parameterized reference inequalities, which will permit analysis of the various subcases, using the indices (i, j), where $i \in \{1, 2\}$ denotes firm i and the derivative variable is X^j, $j \in \{1, 2\}$. These reference inequalities will be shown later for specific ranges of $\tilde{X}$.

$$
\begin{aligned}
\partial_j V_n^i(\tilde{X}) &= 0 = \partial_j V_{n-1}^i(\tilde{X}) && \text{(Eq1)}\\
\partial_j V_n^i(\tilde{X}) &= \mathcal{D}J_n^{i1}(Y^1)|_{Y^1=X^j} + \mathcal{D}J_n^{i2}(Y^2)|_{Y^2=X^j+K_1} \geq 0 = \partial_j V_{n-1}^i(\tilde{X}) && \text{(Eq2)}\\
\partial_j V_n^i(\tilde{X}) &= \mathcal{D}J_n^{i1}(Y^1)|_{Y^1=X^j} + \mathcal{D}J_n^{i2}(Y^2)|_{Y^2=X^j+K_1} \\
&\geq \mathcal{D}J_{n-1}^{i1}(Y^1)|_{Y^1=X^j} + \mathcal{D}J_{n-1}^{i2}(Y^2)|_{Y^2=X^j+K_1} = \partial_j V_{n-1}^i(\tilde{X}) && \text{(Eq3)}\\
\partial_j V_n^i(\tilde{X}) &= \mathcal{D}J_n^{i1}(Y^1)|_{Y^1=X^j} \geq \mathcal{D}J_{n-1}^{i1}(Y^1)|_{Y^1=X^j} \\
&\geq \mathcal{D}J_{n-1}^{i1}(Y^1)|_{Y^1=X^j} + \mathcal{D}J_{n-1}^{i2}(Y^2)|_{Y^2=X^j+K_1} = \partial_j V_{n-1}^i(\tilde{X}) && \text{(Eq4)}\\
\partial_j V_n^i(\tilde{X}) &= \mathcal{D}J_n^{i1}(Y^1)|_{Y^1=X^j} \geq \mathcal{D}J_{n-1}^{i1}(Y^1)|_{Y^1=X^j} = \partial_j V_{n-1}^i(\tilde{X}) && \text{(Eq5)}\\
\partial_j V_n^i(\tilde{X}) &= 0 \geq \partial_j V_{n-1}^i(\tilde{X}) && \text{(Eq6)}\\
\partial_j V_n^i(\tilde{X}) &= \mathcal{D}J_n^{i1}(Y^1)|_{Y^1=X^j} \geq 0 = \partial_j V_{n-1}^i(\tilde{X}) && \text{(Eq7)}
\end{aligned}
$$

While the comparisons are fairly simple, the complicating factor is that different constraints (and corresponding different formula) describe equilibria in various regions. Figure 6 shows why various comparisons are needed and also gives intuition why they hold. It illustrates the dominance of derivatives for adjacent periods, but also shows why the ordering of base-stock levels is crucial. Consider case (A) and its two subcases: (I) $z_{n-1}^1 < z_n^2 - K_1$ and (II) $z_n^2 - K_1 \leq z_{n-1}^1$. Given the needed ordering of thresholds, all of the comparisons are straightforward. For example in Figure 6, derivatives are 0 below $\min(z_{n-1}^1, z_n^1)$, in period n the derivative is positive between z_n^1 and z_{n-1}^1, while still 0 in period $n-1$. The ordering between $\min(z_{n-1}^1, z_n^1)$ and the next threshold follows

	z^1_{n-1}		$z^2_{n-1} - K_1$	
$\partial_1 V^1_{n-1} =$	0	$\mathcal{D}J^{11}_{n-1}(Y^1)\vert_{Y^1=X^1} + \mathcal{D}J^{12}_{n-1}(Y^2)\vert_{Y^2=X^1+K_1}$		$\mathcal{D}J^{11}_{n-1}(Y^1)\vert_{Y^1=X^1}$
$\partial_1 V^1_{n} =$	0	$\mathcal{D}J^{11}_{n}(Y^1)\vert_{Y^1=X^1} + \mathcal{D}J^{12}_{n}(Y^2)\vert_{Y^2=X^1+K_1}$		$\mathcal{D}J^{11}_{n}(Y^1)\vert_{Y^1=X^1}$
	z^1_n		$z^2_n - K_1$	

Figure 6: Derivative dominance in periods n and $n+1$: Case (A)

from the inductional step, etc. Formally, we can relate each case and subcase to the reference inequalities (Eq1)-(Eq7) above:

$(i,j) = (1,1)$: For subcase (I), $X^1 \leq z^1_n \to$ (Eq1), $z^1_n \leq X^1 \leq \min(z^1_{n-1}, z^2_n - K_1) \to$ (Eq2); for subcase (II), $z^2_n - K_1 \leq X^1 \leq z^2_{n-1} - K_1 \to$ (Eq7)

$(i,j) = (2,1)$: For subcases (I) and (II), same as for $(i,j) = (1,1)$ for $X^1 \leq z^2_n - K_1$

$(i,j) = (1,2)$: For subcase (I), $X^2 \leq z^1_n \to$ (Eq3), $z^1_n \leq X^2 \leq z^1_{n-1} \to$ (Eq6), $z^1_{n-1} \leq X^2 \leq z^2_n \to$ (Eq1), $z^2_n \leq X^2 \leq z^2_{n-1} \to$ (Eq7), $z^2_{n-1} \leq X^2 \to$ (Eq5); for subcase (II), $z^1_n \leq X^2 \leq \min(z^1_{n-1}, z^2_{n-1} - K_1, z^2_n) \to$ (Eq6)

$(i,j) = (2,2)$: same as for $(i,j) = (1,2)$

Now, consider case (B) and its two subcases: (I) $z^2_{n-1} - K_1 \leq z^1_{n-1}$ and (II) $z^1_{n-1} < z^2_{n-1} - K_1$. In both these subcases we assume (I) $z^2_{n-1} - K_1 \leq z^1_{n-1} \leq z^2_n$ and (II) $z^1_{n-1} \leq z^2_{n-1} - K_1 \leq z^2_n$. If this is not the case, the analysis reduces to fewer, simpler cases.

$(i,j) = (1,1)$: For subcase (I), $X^1 \leq z^2_n - K_1 \to$ (Eq1), $z^2_n - K_1 \leq X^1 \leq z^2_{n-1} - K_1 \to$ (Eq7), $z^2_{n-1} - K_1 \leq X^1 \to$ (Eq5); for subcase (II), $z^2_n - K_1 \leq X^1 \leq z^1_{n-1} \to$ (Eq7), $z^1_{n-1} \leq X^1 \leq z^2_{n-1} - K_1 \to$ (Eq4)

$(i,j) = (2,1)$: For subcases (I) and (II), $X^1 \leq z^2_n - K_1 \to$ (Eq1)

$(i,j) = (1,2)$: For subcase (I), $X^2 \leq z^2_n - K_1 \to$ (Eq3), $z^2_n - K_1 \leq X^2 \leq z^2_{n-1} - K_1 \to$ (Eq6), $z^2_{n-1} - K_1 \leq X^2 \leq z^2_n \to$ (Eq1); for subcase (II), $z^2_n - K_1 \leq X^1 \leq z^1_{n-1} \to$ (Eq6), $z^1_{n-1} \leq X^1 \leq z^2_n \to$ (Eq1)

$(i,j) = (2,2)$: same as for $(i,j) = (1,2)$, $z^2_n \leq X^2 \leq z^2_{n-1} \to$ (Eq7), $z^2_{n-1} \leq X^2 \to$ (Eq5)

Case (C) has similar logic. Across cases (A), (B), and (C), the following is true $\partial_j V^i_n(\tilde{X}) \geq \partial_j V^i_{n-1}(\tilde{X})$ for $i,j = 1,2$, and limited for $(i,j) = (1,2), X^2 \leq z^2_n$, and $(i,j) = (2,1), X^1 \leq \min(z^2_n - K_1, z^1_n)$. Since we are dealing with the territory *below* the base-stock levels, these dominance conditions also hold for $\mathrm{E}[V(\tilde{Y} - \tilde{D})]$. Multiplying both sides by β and adding L^i to both sides also maintains these conditions, resulting in $\partial_j J^i_{n+1}(\tilde{Y}) \geq \partial_j J^i_n(\tilde{Y})$ for $i,j = 1,2$, and limited for $(i,j) = (1,2), Y^2 \leq z^2_n$, and $(i,j) = (2,1), Y^1 \leq \min(z^2_n - K_1, z^1_n)$. Given the convexity shown for the necessary territory, this is sufficient for (x), which in turn yields (iii). Notice that firm 1's periodic cost (which has a minimum at y^*_{my}) is added to the discounted expected cost-to-go, V^1_n, that is constant in X_1 for $X^1 \leq z^1_n$, since the equilibrium is dependent only upon X^2 (and the equilibrium up-to levels for future periods is higher. Formally, $\partial_1 V^1_n = 0$ for $X^1 \leq \min(z^1_n, z^2_n - K_1)$ in cases (A) and (B) and for $X^1 \leq \underline{z}^1_n$ for case (C). This is preserved under the operation $\beta\mathrm{E}[V^1_n(\tilde{Y} - \tilde{D})]$ and thus $\underline{z}^1_{n+1} = y^*_{my} = \arg\min_{Y^1}[L^1(\tilde{Y}) + \beta\mathrm{E}[V^1_n(\tilde{Y} - \tilde{D})]]$ yielding (xi).

• Independence of the best response outside the band:
We now consider the state space above the band, $Y^2 \geq Y^1 + K_1$. From (xii), J_n^1 is convex in Y^1, decreasing in $Y^1 \leq z_n^1$ for $Y^2 \leq z_n^1 + 2K_1$, independent of Y^2 for $Y^2 \leq z_{n-1}^2$, separable for $Y^1 + K_1 \leq Y^2 \leq Y^1 + 2K_1$, and $J_n^1(Y^1, Y^1 + K_1)$ is decreasing for $Y^1 \leq z_n^1$. Thus, the response function r_n^1 is within the band (for $Y^2 \leq z_n^1 + K_1$ it properly describes the best response function when the supplier is not limited to the band). From (xiii), for $z_n^1 + K_1 \leq X^2 \leq z_n^1 + 2K_1$), the retailer's best response is independent of Y^2, $r_n^1(Y^2) = z_n^1$ (i.e., it will rise "vertically" in the slice $\mathcal{B}^{K_1}$ above the band), and consequently $\mathcal{D}r_n^1(Y^2) = 0$ for $z_n^1 + K_1 \leq Y_2 \leq z_n^1 + 2K_1$).
Now consider $\tilde{X} \in \mathcal{B}^{K_1}$. For notational efficiency, let $z = \min(z_n^1 - K_1, z_n^2 - 2K_1)$. For $X^1 \leq z$, $V_n^1(\tilde{X}) = J_n^1(X^1 + K_1, X^1 + 2K_1)$, adopting the decreasing convexity from the upper edge of the band, $\mathcal{B}_n$; for $z \leq X^1 \leq z+K_1$ and $X^2 \leq z+2K_1$, $V_n^1(\tilde{X}) = J_n^1(z+K_1, z+2K_1)$; for $z+2K_1 \leq X^2 \leq z_n^1+K_1$, $V_n^1(\tilde{X}) = J_n^1(X^2-K_1, X^2)$[3]; for $z_n^1-K_1 \leq X^1 \leq z_n^1$ and $z_n^1+K_1 \leq X^2 \leq z_n^1+2K_1$, $V_n^1(\tilde{X}) = J_n^1(z_n^1, X^2)$; for $X^1 \geq z_n^1$, $V_n^1(\tilde{X}) = J_n^1(X^1, X^2)$. The important element is that the solution follows the upper edge of the band up to $\min(z_n^1, z_n^2 - K_1)$ resulting in the retailer's decreasing convexity in X^1 and a zero slope for both the retailer and supplier with respect to X^2 within $\mathcal{B}^{K_1}$ for $X^2 \leq z+K_1$. Thus, from the assumed convexity of J_n^1 in Y^1, the separability of J_n^1, and $\partial_2 J_{k+1}^1 = 0$, we have V_n^1 is convex decreasing in $X^1 \leq z$, independent of X^1 for $z \leq X^1 \leq z_n^1$, increasing convex for $X^1 \geq z_n^1$, and constant in X^2 for $X^2 \leq z_n^2$. Thus, $V_n^1(\tilde{X}) = V_n^{11}(X^1) + V_n^{12}(X^2)$. These separability and convexity in X_1 properties are preserved for the operation $\beta \mathrm{E}[V_n(\tilde{Y} - \tilde{D})]$. Adding $L^1(\tilde{Y})$ will then yield $J_{n+1}^1(\tilde{Y})$ which inherits the convexity and separability properties. As the minimizer of $J_{n+1}^1(\tilde{Y})$ is z_{n+1}^1, independent of Y^2 for $z_{n+1}^1 + K \leq Y^2 \leq z_{n+1}^1 + 2K_1$, the first claim in (xiii) follows. Consequently convexity and monotonicity of $J_{n+1}^1(\tilde{Y})$ for $Y_1 \leq z_{n+1}^1$ follows. As, V_n^2 has zero slope in X^2 for $X^2 \leq z + K_1$ and positive slope in X^2 for $X^2 > z + K_1$, properties conveyed to J_{n+1}^2. Thus, all properties are inherited yielding (xii) for $\tilde{X} \in \mathcal{B}^{K_1}$.
For $X^2 \geq X^1 + 2K_1$, from convexity of J_n^1 in Y^1 and the definition of r_n^1, we have $V_n^1(\tilde{X}) = J_n^1(X^1 + K_1, X^2)$. Consequently, V_n^1 is convex decreasing in $X^1 \leq z_n^1$ and independent of $X^2 \leq z_n^2$ and convexity in X^1 properties are preserved for the operation $\beta \mathrm{E}[V_n(\tilde{Y} - \tilde{D})]$ and adding L^1. Since convexity holds at $Y^2 = Y^1 + 2K_1$, it holds across all Y^1. Also, the increasing convexity of J_{n+1}^1 for $Y^1 \geq z_{n+1}^1$ implies $r_{n+1}^1(Y^2) \leq Y^2 - 2K_1$ for $Y^2 > z_{n+1}^1 + 2K_1$, which completes (xiii). Consequently, the retailer's best response will not return to the band $\mathcal{B}$ for $Y^2 > z_{n+1}^1 + 2K_1$.

Induction Basis: The salvage value function for player 1 is applied in period 0, while the salvage value function for player 2 is applied in period 1. Mirroring the claims in Lemma A1, for any $\epsilon \in (0, (h_1 + h_2)/4)$, let τ_0 be such that $L'(\tau_0) > h_1 + h_2 - \epsilon$, and τ_1 such that for all $\tau \geq \tau_1$, we have $L'(\tau) - \int_{K_1}^{\infty} L'(\tau + K_1 - D) f(D) dD < \epsilon$. We set (a) $\gamma_1 = \max(y_{my}^*, \tau_0, \tau_1) + \mathrm{E}[D]$ and $\lambda_1 = (h_1 + h_2)/(4\beta \mathrm{E}[D])$, (b) $\gamma_2 = z_1^1$, and (c) $\lambda_2 \geq \frac{\beta p_2 \Pr(D+D>K_1)}{1-\beta}$. This will guarantee the ordering of derivatives below.

Period 1: We now allow k to assume values of 0 and 1, corresponding to periods 1 and 2. In Period 1 we have:

$$
\begin{aligned}
J_1^1(\tilde{Y}) &= \mathrm{E}[(h_1 + h_2)(Y^1 - D)^+ + p_1(D - Y^1)^+] + \beta \lambda_1 \mathrm{E}[(Y^1 - D - \gamma_1)^2] \\
J_1^2(\tilde{Y}) &= h_2(Y^2 - Y^1) + p_2 \mathrm{E}[(D - Y^1)^+]
\end{aligned}
$$

Clearly, $J_1^1(\tilde{Y}) = J_1^{11}(Y^1) + J_1^{12}(Y^2)$, so $J_1^{12}(Y^2) = 0$, and $J_1^2(\tilde{Y}) = J_1^{21}(Y^1) + J_1^{22}(Y^2)$, so $J_1^{21}(Y^1) = -h_2 Y^1 + p_2 \mathrm{E}[(D - Y^1)^+]$ and $J_1^{22}(Y^2) = h_2 Y^2$. Since J_1^i are continuous in $\tilde{Y}$, convex in Y^i, and the action set, $\mathcal{A}(\tilde{X})$, is nonempty compact convex subset of a Euclidean space (as in the inductional

[3]This case exists only when $z_n^2 - K_1 \leq z_n^1$. Moreover, this is the distorting case where the slope of the retailer's best response for $X^2 \geq X^1 + 2K_1$ may be increased from zero over multiple periods.

step), from Theorem 1 there exists at least one pure-strategy Nash equilibrium, which occurs at the intersection of best-reply functions. Due to the strict convexity of J_1^1 with respect to Y^1 and the strict monotonicity of J_1^2, the equilibrium is unique in period 1.

Denote $z_1^1 := \arg\min_{Y^1} J_1^1(\tilde{Y})$ and $z_1^2 := \arg\min_{Y^2} J_1^2(\tilde{Y}) = -\infty$. The equilibrium is:

$$Y^1 = z^1|[X^1, X^2] \text{ and } Y^2 = X^2$$

yielding (i) for $n = 1$. Based on Karush's (1959) lemma $V_1^i(\tilde{X}) = V_1^{i1}(X^1) + V_1^{i2}(X^2)$ since J_1^i is separable for $i = 1, 2$, establishing (ii). For (iv)-(vii), it is sufficient to show the desired properties for V_1^i. Note that Lemma 1 permits us to restrict attention to $\tilde{X} \in \mathcal{B}$ only. From the definitions of J_1^i (see above), the period 1 equilibrium, and the separability, we have that V_1^1 and V_1^2 are flat in $X^1 \leq z_1^1$ and convex increasing and convex decreasing, respectively, in $X^1 \geq z_1^1$ yielding (iv) and (v). $V_1^1(\tilde{X}) = J_1^1(X^2, X^2) = J_1^{11}(X^2)$ is convex decreasing in X^2 for $X^2 \leq z_1^1$ since J_1^{11} is convex and minimized at z_1^1. $V_1^2(\tilde{X}) = J_1^2(X^2, X^2) = p_2\text{E}[(D - X^2)^+]$ is convex decreasing in X^2 for $X^2 \leq z_1^1$. $V_1^i(\tilde{X}) = J_1^i(z_1^1|[X^1, X^2], X^2) = J_1^{i1}(\max(z_1^1, X^1)) + J_1^{i2}(X^2)$ for $X^2 \geq z_1^1$. Thus, $V_1^1(\tilde{X}) = J_1^{11}(\max(z_1^1, X^1))$ so $\partial_2 V_1^1 = 0$ and $V_1^2(\tilde{X}) = J_1^{21}(\max(z_1^1, X^1)) + J_1^{22}(X^2) = J_1^{21}(\max(z_1^1, X^1)) + h_2 X^2$ so $\partial_2 V_1^2 = h_2 > 0$ for $X^2 \geq z_1^1$. This means that, V_1^i is convex in X^i and non-increasing in X^{-i} yielding (v) and (vi). While in any period $n \geq 2$, the analysis within band $\mathcal{B}$ and outside this band will differ, for period 1 the slope (derivative) of V_1^1 with respect to X^1 is constant in X^2. Thus, the derived best reply function in period 2, r_2^1, will be completely vertical along the Y^2 dimension, thus justifying (xiii). Likewise, (xii) is justified since J_1^1 is convex in Y^1 around z_1^1 and J_1^2 is increasing in Y^2.

<u>*Period 2:*</u> From the definition of the salvage value functions above, $\partial_1 S_1^2(\tilde{X}) = 0$ and

$$\partial_2 S_1^2(\tilde{X}) = \begin{cases} -\lambda_2 & X^2 \leq \gamma_2 \\ 0 & \gamma_2 \leq X^2 \end{cases}$$

so S_1^2 is convex non-increasing in X^2 and independent of X^1. Thus, $S_1^2 + V_1^2$ is convex in X^2 and non-increasing in X^1. Consequently, $\beta\text{E}[S_1^2(\tilde{Y} - \tilde{D})] + \beta\text{E}[V_1^2(\tilde{Y} - \tilde{D})] + L^2(\tilde{Y}) =: J_2^2(\tilde{Y})$ has the same properties, yielding (vi) and (vii). Condition (3) in the main paper ensures a finite minimizer with respect to Y^2. Likewise, $V_1^1(\tilde{Y})$ is convex in Y^1 and convex non-increasing in Y^2, implying that $J_2^1(\tilde{Y}) := L^1(\tilde{Y}) + \beta\text{E}[V_1^1(\tilde{Y} - \tilde{D})]$ has these same properties, yielding (iv) and (v). This establishes the basis convexity conditions. Specifically, $\partial_1 V_1^1 = 0$ for $Y^1 \leq z_1^1$ and convex non-decreasing for $Y^1 > z_1^1$, which remains true for $\beta\text{E}[V_1^1(\tilde{Y} - \tilde{D})]$. Thus, $\underline{z}_2^1 = \arg\min_{Y^1} J_2^1 = y_{my}^*$, establishing (xii). J_2^i is clearly separable since L^i is separable.

Condition (a) is sufficient to establish that $z_2^1 \leq z_1^1$, partly showing (iii), since the upper edge of the band for $Y^1 \leq z_1^1$ mimicks the single dimensional state in the single echelon capacitated problem described in Lemma A1, under identical salvage value conditions. This will then establish the slope of $J_2^1(Y^1, Y^1 + K_1)$ at $(z_1^1, z_1^1 + K_1)$ will be non-negative, sufficient to establish (ix) for the retailer, and combined with the convexity results, sufficient to show (x) for the retailer. It is straightforward to see the equilibrium solution is $Y_2^1 = \max(\underline{z}_2^1, \min(z_2^1, z_2^2 - K_1))|[X^1, X^2]$ and $Y_2^2 = z_2^2|[X^2, Y_2^1 + K_1]$.

To establish the basis for $z_{n+1}^2 \leq z_n^2$, however, we need to consider an additional period since $z_1^2 = -\infty$. Clearly, there exists the freedom to choose values of γ_1 and γ_2 to achieve $z_2^2 \leq z_1^1$, which we do (partially establishing (iii)).

We consider the same three cases: (A) $z_2^1 \leq z_2^2 - K_1$, (B) $\underline{z}_2^1 < z_2^2 - K_1 \leq z_2^1$, and (C) $z_2^2 - K_1 \leq \underline{z}_2^1 \leq z_2^2$. Consider first (A). Using the derived equilibrium solution (Y^1, Y^2) above, we have

$$\partial_2 V_2^2(\tilde{X}) = \partial_1 J_2^2(\tilde{Y})|_{Y^1=X^2} + \partial_2 J_2^2(\tilde{Y})|_{Y^2=X^2+K_1}. \tag{Eq8}$$

For $X^2 \le z_1^1$, $\partial_2 V_1^2(\tilde{X}) = \mathcal{D}[p_2 \mathrm{E}[(D-X^2)^+]] < 0$; for $X^2 \ge z_1^1$, $\partial_2 V_1^2(\tilde{X}) = h_2 > 0$. Thus, V_1^2 has a finite minimizer in X^2 and $V_1^2(\tilde{X}) + S_1^2(\tilde{X})$ has a finite minimizing point with respect to X^2 due to the functional shape of S_1^2. So J_2^2 is convex in Y^2 around $z_2^2 \le z_1^1$,

$$\begin{aligned} J_2^2(\tilde{Y}) &= L^2(\tilde{Y}) + \beta \mathrm{E}[V_1^2(\tilde{Y}-\tilde{D})] + \beta \mathrm{E}[S_1^2(\tilde{Y}-\tilde{D})] \\ &= h_2(Y^2 - Y^1) + p_2 \mathrm{E}[(D-Y^1)^+] + \beta p_2 \mathrm{E}[(D+D-Y^2)^+] + \beta\lambda_2 \mathrm{E}[(D+\gamma_2 - Y^2)^+]. \end{aligned}$$

Thus, $\partial_1 J_2^2 = \partial_1 J_1^2$, partly showing (x). For $X^2 \le z_2^1$, $\partial_2 V_2^2(\tilde{X}) = \mathcal{D}[p_2 \mathrm{E}[(D-Y^1)^+]]|_{Y^1=X^2} + \partial_2 \beta \mathrm{E}[V_1^2(\tilde{Y}-\tilde{D})]|_{Y^2=X^2+K_1} + \partial_2 \beta \mathrm{E}[S_1^2(\tilde{Y}-\tilde{D})]|_{Y^2=X^2+K_1} = -p_2 \Pr(D > X^2) - \beta p_2 \Pr(D > X^2 + K_1 - D) - \beta\lambda_2$, and for $X^2 \le \gamma_2$, $\partial_2[V_1^2(\tilde{X}) + S_1^2(\tilde{X})] = \mathcal{D} p_2 \mathrm{E}[(D-Y^1)^+]|_{Y^1=X^2} - \lambda_2 = -p_2 \Pr(D > X^2) - \lambda_2$.
Since $z_2^1 \le z_1^1 = \gamma_2$, from condition (b) and the expression for $X^2 \le z_2^1$,

$$\begin{aligned} \partial_2 V_2^2(\tilde{X}) &= -p_2 \Pr(D > X^2) - \beta p_2 \Pr(D > X^2 + K_1 - D) - \beta\lambda_2 \\ &\ge -p_2 \Pr(D > X^2) - \lambda_2 = \partial_2[V_1^2(\tilde{X}) + S_1^2(\tilde{X})] \end{aligned}$$

since we assume condition (c) of the theorem statement holds. To see condition (c) is sufficient,

$$\lambda_2 \ge \frac{\beta p_2 \Pr(D + D > K_1)}{1-\beta} \ge \frac{\beta p_2 \Pr(D + D > X^2 + K_1)}{1-\beta}.$$

In case (B), in period 2, for $X^2 \le z_2^2 - K_1$, $\partial_2 V_2^2(\tilde{X}) = \mathcal{D} p_2 \mathrm{E}[(D-Y^1)^+]|_{Y^1=X^2} + \partial_2 \beta \mathrm{E}[V_1^2(\tilde{Y} - \tilde{D})]|_{Y^2=X^2+K_1} + \partial_2 \beta \mathrm{E}[S_1^2(\tilde{Y}-\tilde{D})]|_{Y^2=X^2+K_1}$. In case (C), for $X^2 \le z_2^2 - K_1$, $\partial_2 V_2^2(\tilde{X}) = -p_2 \Pr(D > X^2) - \beta p_2 \Pr(D + D > X^2 + K_1) - \beta\lambda_2 \ge -p_2 \Pr(D > X^2) - \lambda_2 = \partial_2[V_1^2(\tilde{X}) + S_1^2(\tilde{X})]$ and for $z_2^2 - K_1 \le X^2 \le z_2^2$, $\partial_2 V_2^2(\tilde{X}) = 0 \ge -p_2 \Pr(D > X^2) - \lambda_2 = \partial_2[V_1^2(\tilde{X}) + S_1^2(\tilde{X})]$.

For (A), (B), and (C), for $X^2 \le \max(z_2^1, \min(z_2^1, z_2^2 - K_1))$

$$\begin{aligned} \partial_2 V_2^2(\tilde{X}) &\ge \partial_2[V_1^2(\tilde{X}) + S_1^2(\tilde{X})] \\ \partial_2 \mathrm{E}[V_2^2(\tilde{Y}-\tilde{D})] &\ge \partial_2 \mathrm{E}[V_1^2(\tilde{Y}-\tilde{D})] + \partial_2 \mathrm{E}[S_1^2(\tilde{Y}-\tilde{D})] \\ \partial_2[L^2(\tilde{Y}) + \beta \mathrm{E}[V_2^2(\tilde{Y}-\tilde{D})]] &\ge \partial_2[L^2(\tilde{Y}) + \beta \mathrm{E}[V_1^2(\tilde{Y}-\tilde{D})] + \beta \mathrm{E}[S_1^2(\tilde{Y}-\tilde{D})]] \\ \partial_2 J_3^2(\tilde{Y}) &\ge \partial_2 J_2^2(\tilde{Y}) \text{ for } Y^2 \le z_2^1. \end{aligned}$$

For $\min(z_2^1, z_2^2 - K_1) \le X^2 \le z_2^2$,

$$\partial_2 V_2^2(\tilde{X}) = 0 \ge \partial_2 V_1^2(\tilde{X}) + \partial_2 S_1^2(\tilde{X})$$

and following the same steps, we achieve

$$\partial_2 J_3^2(\tilde{Y}) \ge \partial_2 J_2^2(\tilde{Y})$$

showing the final part of (x), implying $z_3^2 \le z_2^2$, yielding (iii), due to the convexity already shown. The analysis deriving the properties for firm 1's best reply function outside the band is very similar to that in the induction step, but slightly simpler since $r_2^1(Y^2)$ is completely straight for values of $Y^2 \ge Y^1 + K_1$, above the band $\mathcal{B}$. This completes the basis. ■

Theorem 3 *If $p_2 \leq h_2(1-\beta)/\beta$, for each starting inventory $\tilde{X} \in \mathcal{B}$, $S_0^1(\tilde{X}) = 0$ and $S_1^2(\tilde{X}) = 0$, there exists a unique pure-strategy Nash equilibrium where the retailer orders up to a myopic base-stock level, y^*_{my}, if possible, and the supplier orders no goods at all. Also, $V_n^i(\tilde{X}) = V_n^{i1}(X^1) + V_n^{i2}(X^2)$ for $i = 1, 2$, and $n > 0$.*

Proof In period 1, $J_1^i(\tilde{Y}) = J_1^{i1}(Y^1) + J_1^{i2}(Y^2)$, $\partial_j \partial_j J_1^i \geq 0$ for $i, j = 1, 2$, $y^*_{my} = \arg\min_{Y^1} J_1^1$, $\partial_2 J_1^1 = 0$, $\partial_1 J_1^2 = -h_2 - p_2 \Pr(D > Y^1) < 0$, and $\partial_2 J_1^2 = h_2 > 0$. The constrained best-reply functions are $r_1^1(Y^2) = y^*_{my}|[X^1, X^2]$ and $r_1^2(Y^1) = X^2$, creating a unique equilibrium. Thus, $V_1^i(\tilde{X}) = J_1^i(y^*_{my}|[X^1, X^2], X^2)$.

Assume: $J_{n-1}^i(\tilde{Y}) = J_{n-1}^{i1}(Y^1) + J_{n-1}^{i2}(Y^2)$ for $i = 1, 2$, $\partial_j \partial_j J_{n-1}^i \geq 0$ for $i = j$ and $i = 1, j = 2$, $y^*_{my} = \arg\min_{Y^1} J_{n-1}^1$, $\partial_2 J_{n-1}^1 \leq 0$, $\partial_1 J_{n-1}^2 \leq 0$, $\partial_1 \partial_1 J_{n-1}^2 \geq 0$ for $Y^1 \leq y^*_{my}$, $\partial_2 J_{n-1}^2 \geq h_2 - p_2 \sum_{i=1}^{n-2} \beta^i$. The constrained best-reply functions are $r_{n-1}^1(Y^2) = y^*_{my}|[X^1, X^2]$ and $r_{n-1}^2(Y^1) = X^2$, creating a unique equilibrium. Thus, $V_{n-1}^i(\tilde{X}) = J_{n-1}^i(y^*_{my}|[X^1, X^2], X^2)$ and is separable since J_{n-1}^i is separable. It is clear $\partial_1 V_{n-1}^i = 0$ for $X^1 \leq y^*_{my}$ for $i = 1, 2$, $\partial_1 V_{n-1}^1 \geq 0$, $\partial_1 \partial_1 V_{n-1}^1 \geq 0$, $\partial_2 V_{n-1}^1 \leq 0$, $\partial_2 \partial_2 V_{n-1}^1 \geq 0$, $\partial_1 V_{n-1}^2 \leq 0$ for $Y^1 > y^*_{my}$, $\partial_2 V_{n-1}^2 \geq -p_2 \Pr(D > Y^2) - p_2 \sum_{i=1}^{n-2} \beta^i \geq -p_2 - p_2 \sum_{i=1}^{n-2} \beta^i = -p_2 \sum_{i=0}^{n-2} \beta^i$, $\partial_2 \partial_2 V_{n-1}^2 \geq 0$. These relationships will be preserved for $\mathrm{E}[V_{n-1}(\tilde{Y} - \tilde{D})]$ and specifically $\partial_2 \mathrm{E}[V_{n-1}^2] \geq -p_2 \sum_{i=0}^{n-2} \beta^i$. Multiplying by β and adding the periodic cost generates J_n^i which have all the requisite properties and specifically, $\partial_2 J_n^2 = h_2 + \beta \partial_2 \mathrm{E}[V_{n-1}^2] \geq h_2 - p_2 \sum_{i=1}^{n-1} \beta^i \geq h_2 - \frac{\beta p_2}{1-\beta} \geq 0$ from the condition in the theorem. ■

Theorem 4 *The equilibrium up-to levels are non-increasing when K_1, h_1, or h_2 increase or when p_1 or p_2 decrease.*

Proof We partition the analysis between that for the economic parameters (h_1, h_2, p_1, and p_2) and the capacity (K_1), primarily in the basis for the induction. Firstly, consider the economic parameters and the two conditions: (a) $p_2 > h_2(1-\beta)/\beta$ and (b) $p_2 \leq h_2(1-\beta)/\beta$. Let us consider changes in the cost parameters, one at a time, but which do not change the status of condition of (a) or (b) for a particular instance of the model (we examine the situation where this is not the case later). Let us consider models satisfying condition (b) first. If condition (b) is maintained for the values described, then the results of Theorem 3 are true. Theorem 3 shows that player 2 orders nothing ($z_n^2 = -\infty$) and player 1 orders up to y^*_{my}, if possible. Since $y^*_{my} := \arg\min_{Y^1} L^1(\tilde{Y})$, increasing h_1 or h_2 or decreasing p_1 results in a non-increase in y^*_{my}, while p_2 has no affect upon y^*_{my}. We will make use of the convexity and separability results of Theorem 2 and the best-response construction results of Lemma 2.

<u>Basis for Induction</u>

Let us now consider models satisfying condition (a). Consider the economic parameters (h_1, h_2, p_1, p_2) first. (Salvage value functions are the same for the different values of these parameters.) We consider the case for h_1 in depth and describe briefly how the other three cases differ slightly:

$$
\begin{aligned}
J_1^1(\tilde{Y}) &= \mathrm{E}[(h_1 + h_2)(Y^1 - D)^+ + p_1(D - Y^1)^+] + \beta\lambda_1 \mathrm{E}[(Y^1 - D - \gamma_1)^2] \\
J_1^2(\tilde{Y}) &= h_2(Y^2 - Y^1) + p_2 \mathrm{E}[(D - Y^1)^+].
\end{aligned}
$$

It is clear that $\partial[\partial_1 J_1^1(\tilde{Y})]/\partial h_1 = \Pr(D \leq Y^1) \geq 0$ for all Y^1. In addition, $\partial[\partial_2 J_1^2(\tilde{Y})]/\partial h_1 = 0$, $\partial[\partial_2 J_1^1(\tilde{Y})]/\partial h_1 = 0$, and $\partial[\partial_1 J_1^2(\tilde{Y})]/\partial h_1 = 0$. Since $J_1^1(\tilde{Y})$ is independent of Y^2, $z_1^1 = \arg\min_{Y^1} J_1^1(\tilde{Y})$, $\partial[\partial_1 J_1^1(\tilde{Y})]/\partial h_1 \geq 0$ and the convexity of J_1^1 results in $\partial z_1^1/\partial h_1 \leq 0$.

Now consider the basis for the capacity. Clearly, $\partial[\partial_1 J_1^1(\tilde{Y})]/\partial K_1 = \partial[\partial_2 J_1^{2B}(\tilde{Y})]/\partial K_1 = \partial[\partial_2 J_1^{1B}(\tilde{Y})]/\partial K_1 = \partial[\partial_1 J_1^{2B}(\tilde{Y})]/\partial K_1 = 0$, implying $\partial z_1^1/\partial K_1 = 0$.

Induction Step

Now for period n, $\partial[\partial_1 J_n^1(\tilde{Y})]/\partial h_1 \geq 0$, $\partial[\partial_2 J_n^1(\tilde{Y})]/\partial h_1 \geq 0$ for all $Y^2 \leq z_n^2$, $\partial[\partial_2 J_n^2(\tilde{Y})]/\partial h_1 \geq 0$, and $\partial[\partial_1 J_n^2(\tilde{Y})]/\partial h_1 \geq 0$ for all $Y^1 \leq \min(z_n^2 - K_1, z_n^1)$. (The analysis will be performed for h_1 but is very similar for the other economic parameters and capacity.)

From the convexity and separability of J_n^2, and $\partial[\partial_2 J_n^2(\tilde{Y})]/\partial h_1 \geq 0$ immediately implies $\partial z_n^2/\partial h_1 \leq 0$ and $\partial r_n^2/\partial h_1 \leq 0$, which has the structure described in Lemma 2. Likewise, from the convexity and separability of J_n^1, $\partial[\partial_1 J_n^1(\tilde{Y})]/\partial h_1 \geq 0$, $\partial[\partial_2 J_n^1(\tilde{Y})]/\partial h_1 \geq 0$, then $\partial \underline{z}_n^1/\partial h_1 \leq 0$, $\partial z_n^1/\partial h_1 \leq 0$, and $\partial r_n^1/\partial h_1 \leq 0$. The analysis with respect to K_1 is similar but since the definition of z_n^1 depends on K_1 (i.e., $z_n^1 = \arg\min_{Y^1} J_n^1(Y^1, Y^1 + K_1)$), we must account for this slight departure from the analysis with respect to h_1. Now, $\partial[\mathcal{D}J_n^1(Y^1, Y^1 + K_1)]_{(Y^1,Y^1+K_1)}/\partial K_1 = \partial[\partial_1 J_n^1(\tilde{Y})]/\partial K_1 + \partial[\partial_2 J_n^1(\tilde{Y})|_{Y^2=Y^1+K_1}]/\partial K_1 = \partial[\partial_1 J_n^1(\tilde{Y})]/\partial K_1 + \partial[\partial_2 J_n^1(\tilde{Y})]/\partial K_1 \partial(Y^1+K_1)/\partial K_1 \geq 0$. Thus, $\partial z_n^1/\partial K_1 \leq 0$. From Theorem 2, the convexity of J_n^i is mapped to V_n^i via the equilibrium solution: $\partial[\partial_1 V_n^1(\tilde{Y})]/\partial h_1 \geq 0$, $\partial[\partial_2 V_n^1(\tilde{Y})]/\partial h_1 \geq 0$ for all $Y^2 \leq z_n^2$, $\partial[\partial_2 V_n^2(\tilde{Y})]/\partial h_1 \geq 0$, and $\partial[\partial_1 V_n^2(\tilde{Y})]/\partial h_1 \geq 0$ for all $Y^1 \leq \min(z_n^2 - K_1, z_n^1)$. For example, a typical solution in period n for $X^2 \leq \min(z_n^1, z_n^2 - K_1)$, $V_n^i(\tilde{X}) = J_n^i(X^2, X^2 + K_1)$ and due to separability and the induction assumption, $\partial[\partial_2 V_n^i(\tilde{X})]/\partial h_1 = 0$ and $\partial_2 V_n^i(\tilde{X}) = \mathcal{D}J_n^{i1}(Y)|_{Y=X^2} + \mathcal{D}J_n^{i2}(Y)|_{Y=X^2+K_1} \geq \mathcal{D}J_n^{i1}(Y)|_{Y=X^2} + \mathcal{D}J_n^{i2}(Y)|_{Y=X^2}$ and so $\partial[\partial_2 V_n^i(\tilde{X})]/\partial h_1 = \partial[\mathcal{D}J_n^{i1}(X^2)]/\partial h_1 + \partial[\mathcal{D}J_n^{i2}(X^2 + K_1)]/\partial h_1 \geq 0$. Consequently, $\partial[\partial_j \mathrm{E}[V_n^i(\tilde{Y} - \tilde{D})]]/\partial h_1 \geq 0$ over the corresponding domains. Since $\partial[\partial_1 L^1(\tilde{Y})]/\partial h_1$, $\partial[\partial_1 J_{n+1}^1(\tilde{Y})]/\partial h_1 \geq 0$, $\partial[\partial_2 J_{n+1}^1(\tilde{Y})]/\partial h_1 \geq 0$ for all $Y^2 \leq z_n^2$, $\partial[\partial_2 J_{n+1}^2(\tilde{Y})] \geq 0$, and $\partial[\partial_1 J_{n+1}^2(\tilde{Y})]/\partial h_1 \geq 0$ for all $Y^1 \leq \min(z_n^2 - K_1, z_n^1)$, completing the induction. This establishes that increasing h_1 has a non-increasing effect upon z_n^1 and z_n^2 under condition (a). For the cases of increasing h_2 or K_1 or decreasing either p_1 or p_2, the steps of analysis are very similar to that of h_1

What about a change in the cost parameters p_2 and h_2, whereby there is a move from condition (a) to condition (b)? (The discussion will be identical for the converse case.) Theorem 3, where condition (b) holds, establishes that $z_n^2 = -\infty$ and $z_n^1 = y_{my}^*$. Theorem 2, where condition (a) holds, establishes some z_n^1 which is greater than or equal to y_{my}^*, and a finite value of z_n^2. Therefore, a decrease of p_2, such as (iv), causing a change from condition (a) to condition (b), results in change from a (potentially) finite value of z_n^2 to $z_n^2 = -\infty$ and a change from a value of $z_n^1 > y_{my}^*$ to $z_n^1 = y_{my}^*$. Since the definition of the holding costs is as echelon holding costs, an increase in h_2 will result in a decrease in the myopic order-up-to level from the definition of the periodic costs of player 1. Changes in h_1 or p_1 will not cause a switch from condition (a) to (b) or vice versa. ■

References

[1] Karush, W., "A Theorem in Convex Programming," *Naval Research Logistics Quarterly*, Vol. 6, No. 3 (1959) 245-260.