On Hospice Operations under Medicare Reimbursement Policies: Appendix

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Appendix: Not Intended for Print Publication

A Proofs

Proof of Proposition 1. Expected patient revenue equals $\mathrm{E}[R \wedge \kappa]$, where $R \sim rN(m, \sigma^2)$ and $\kappa = (\lambda_1 + \lambda_2)K$. Then, $\mathrm{E}[R \wedge \kappa] = r\sigma \mathrm{E}[Z \wedge z] + rm$ where Z is the standard normal r.v. and $z = (\kappa - rm)/(r\sigma)$. Note that $\mathrm{E}[Z \wedge z] = -\phi(z) + z[1 - \Phi(z)]$ which follows directly from the properties of the normal distribution (e.g., Zipkin, 2000, p.459). Then, substituting $\bar{m} = m/\lambda$, $\bar{\sigma}^2 = \sigma^2/\lambda$, and $\kappa = \lambda K$ and rearranging terms yields total expected revenue as

$$\mathrm{E}[R \wedge \kappa] = \lambda \left[K - (K - r\bar{m})\Phi\left(\frac{K/r - \bar{m}}{\bar{\sigma}/\sqrt{\lambda}}\right) - \frac{r\bar{\sigma}}{\sqrt{\lambda}}\phi\left(\frac{K/r - \bar{m}}{\bar{\sigma}/\sqrt{\lambda}}\right)\right].$$

The expected cost is $c_1 m_1 \lambda_1 + c_2 m_2 \lambda_2 + A = \bar{c}\lambda + A$ and the result follows.

For the comparative static results,

$$\frac{\partial \pi}{\partial K} = -\frac{1}{2}\lambda\left(\frac{2}{\sqrt{\pi}}\int_0^{\tilde{z}} e^{-t^2} dt - 1\right) \text{ where } \tilde{z} = \frac{\lambda(K - r\bar{m})}{\sqrt{2}r\bar{\sigma}}.$$

Because $\frac{2}{\sqrt{\pi}}\int_0^{\tilde{z}} e^{-t^2} dt$ is the Gaussian error function, bounded above by 1, $\frac{\partial \pi}{\partial K} > 0$. Moreover, some additional algebra shows $\frac{\partial^2 \pi}{\partial K^2} < 0$. So, π is concave increasing in K. Further,

$$\frac{\partial^2 \pi}{\partial r^2} = -\frac{e^{-\frac{\lambda(K - r\bar{m})^2}{2r^2\bar{\sigma}^2}} K^2 \lambda^{3/2}}{\sqrt{2\pi} r^3 \bar{\sigma}} < 0, \qquad \frac{\partial \pi}{\partial \sigma_1} = -\frac{\lambda_1 e^{-\frac{\lambda(K - r\bar{m})^2}{2r^2\bar{\sigma}^2}} r\sigma_1}{\sqrt{2\pi\lambda}\bar{\sigma}} < 0, \qquad \frac{\partial \pi}{\partial \sigma_2} = -\frac{\lambda_2 e^{-\frac{\lambda(K - r\bar{m})^2}{2r^2\bar{\sigma}^2}} r\sigma_2}{\sqrt{2\pi\lambda}\bar{\sigma}} < 0,$$

$$\frac{\partial \pi}{\partial c_1} = -\lambda_1 m_1 < 0, \qquad \frac{\partial \pi}{\partial c_2} = -\lambda_2 m_2 < 0, \qquad \frac{\partial \pi}{\partial A} = -1 < 0.$$

Thus, π is concave in r, decreasing in σ_1 and σ_2, and linearly decreasing in c_1, c_2, and A. ■

Proof of Proposition 2. Fix t, i, and j, and consider one unit of class i, j fluid arriving at time t, the fraction remaining in the system at time $s \geq t$ equals $\Pr(Y_i^j > s - t)$, where Y_i^j is a generic random variable with density $f_i^j(\cdot)$. Note that lifetimes are not random in this model so $\Pr(Y_i^j > s - t)$ is the exact fraction that remains, not the expected fraction. Then the terminal value $v_i^j(t)$ associated with one unit of class i, j fluid arriving at time t is given by

$$v_i^j(t) = v_i^j \Pr(Y_i^j > T - t) = v_i^j \int_{T-t}^{\infty} f_i^j(x) dx.$$

The cumulative (potential) revenue $r_i^j(t)$ over $[0, T]$ from one unit of class i, j fluid arriving at time t is given by

$$r_i^j(t) = r\int_t^T \Pr(Y_i^j > s - t) ds = r\int_0^{T-t} \Pr(Y_i^j > x) dx = r\int_0^{\infty} (x \wedge (T - t)) f_i^j(x) dx,$$

where the final equality is achieved with an interchange of integrals. The derivation of $c_i^j(\cdot)$ follows similarly. ■

Proof of Proposition 3.

To simplify the analysis, we define for all i, t, q,

$$\begin{aligned}
\delta_i(t;q) &= K + (r_i(t) - K)q - c_i(t) + v_i(t), \\
\tilde{\delta}_i(t;q) &= -\tilde{r}_i(t)q + \tilde{c}_i(t) - \tilde{v}_i(t),
\end{aligned}$$

and make the following technical assumption.

Technical Assumption 1: The number of times the functions $\delta_i(\cdot;q)$ and $\tilde{\delta}_i(\cdot;q)$ change sign on $[0,T]$ is uniformly bounded for all $q \in [0,1]$.

This is a reasonable assumption given that $(r_i(\cdot), c_i(\cdot), v_i(\cdot))$ and $(\tilde{r}_i(\cdot), \tilde{c}_i(\cdot), \tilde{v}_i(\cdot))$ are smooth functions. Moreover, it is satisfied when the length of stay distribution is exponential or gamma.

By the technical assumption, there exists N, and for $i = 1, 2$, continuous trigger functions $\{\underline{t}_{ij}(\cdot), \bar{t}_{ij}(\cdot)\}_{j=1}^N$ and $\{\underline{\tau}_{ij}(\cdot), \bar{\tau}_{ij}(\cdot)\}_{j=1}^N$ corresponding to the points at which $\delta_i(\cdot;q)$ and $\tilde{\delta}_i(\cdot;q)$ change sign, respectively, such that for all i, q

$$0 \le \underline{t}_{i1}(q) \le \bar{t}_{i1}(q) \le \ldots \le \underline{t}_{iN}(q) \le \bar{t}_{iN}(q) \le T,$$

$$0 \le \underline{\tau}_{i1}(q) \le \bar{\tau}_{i1}(q) \le \ldots \le \underline{\tau}_{iN}(q) \le \bar{\tau}_{iN}(q) \le T,$$

which are defined as follows. We will only construct $\{\underline{\tau}_{ij}(\cdot), \bar{\tau}_{ij}(\cdot)\}_{j=1}^N$; and the construction of $\{\underline{t}_{ij}(\cdot), \bar{t}_{ij}(\cdot)\}$ follows similarly.

To this end, fix $i = 1, 2$ and $q \in [0,1]$. If $\tilde{\delta}_i(\cdot;q)$ is always positive, then let $\underline{\tau}_{i1}(q) = 0, \bar{\tau}_{i1}(q) = T$ and $\underline{\tau}_{ij}(q) = \bar{\tau}_{ij}(q) = T$ for $j = 2, \ldots, N$. If $\tilde{\delta}_i(\cdot;q)$ is always non-positive, then let $\underline{\tau}_{ij}(q) = \bar{\tau}_{ij}(q) = T$ for all j. Otherwise, $\tilde{\delta}_i(\cdot;q)$ changes sign at least once. Let $0 < \gamma_i^1 < \ldots < \gamma_i^M < T$ denote the times at which $\tilde{\delta}_i(\cdot;q)$ changes sign, where $1 \le M \le 2N$. For notational convenience, let $\gamma_i^0 = 0$ and $\gamma_i^{M+1} = T$. There are two cases to consider:

Case i: The sign switches to positive at γ_i^1. Then

$$(\underline{\tau}_{ij}(q), \bar{\tau}_{ij}(q)) = \begin{cases} (\gamma_i^{2j-1}, \gamma_i^{2j}) & \text{for } 1 \le j \le \lceil \frac{M}{2} \rceil, \\ (T, T) & \text{otherwise.} \end{cases}$$

Case ii: The sign switches to negative at γ_i^1. Then

$$(\underline{\tau}_{ij}(q), \bar{\tau}_{ij}(q)) = \begin{cases} (\gamma_i^{2(j-1)}, \gamma_i^{2j-1}) & \text{for } 1 \le j \le \lfloor \frac{M}{2} + 1 \rfloor, \\ (T, T) & \text{otherwise.} \end{cases}$$

The trigger functions $\underline{t}_{ij}(\cdot)$ and $\bar{t}_{ij}(\cdot)$ are defined similarly. Moreover, assumption (5) ensures that for every q and some i, $0 \leq \underline{t}_{i1}(\cdot) < \bar{t}_{i1}(\cdot) < T$. Given the trigger functions, we write

$$
\begin{aligned}
F(q) \quad = \quad & -K(\lambda_1+\lambda_2)T + R(0) + \sum_{i=1}^{2}\lambda_i \int_0^T r_i(s)ds \\
& + \sum_{i=1}^{2}\sum_{j=1}^{N}\int_{\underline{t}_{ij}(q)}^{\bar{t}_{ij}(q)} \frac{r_i(t)-K}{\eta_i^s}(K+(r_i(t)-K)q - c_i(t) + v_i(t))dt \\
& + \sum_{i=1}^{2}\sum_{j=1}^{N}\int_{\underline{\tau}_{ij}(q)}^{\bar{\tau}_{ij}(q)} \frac{-\tilde{r}_i(t)}{\eta_i^l}(-\tilde{r}_i(t)q + \tilde{c}_i(t) - \tilde{v}_i(t))dt.
\end{aligned}
$$

Note that the trigger functions may not be differentiable at all points. However, it is straightforward to argue that their right and left derivatives exist. Therefore, a viable proof strategy to show differentiability is to establish that the right and left-derivatives of F are equal. In the interest of brevity, we will proceed as if the trigger functions are differentiable, but it will be clear in calculating F' that whether we use the left or the right derivatives makes no difference in calculating F' because the terms involving $\underline{t}'_{ij}, \bar{t}'_{ij}, \underline{\tau}'_{ij}, \bar{\tau}'_{ij}$ will vanish.

To be more specific, note by Leibnitz differentiation rule that

$$
\begin{aligned}
F'(q) \quad = \quad & \sum_{i=1}^{2}\sum_{j=1}^{N}\left[\int_{\underline{t}_{ij}(q)}^{\bar{t}_{ij}(q)} \frac{(r_i(t)-K)^2}{\eta_i^s}dt + \int_{\underline{\tau}_{ij}(q)}^{\bar{\tau}_{ij}(q)} \frac{(\tilde{r}_i(t))^2}{\eta_i^l}dt\right] \\
& + \sum_{i=1}^{2}\sum_{j=1}^{N}\left[\frac{r_i(\bar{t}_{ij}(q))-K}{\eta_i^s}\delta_i(\bar{t}_{ij}(q);q)\bar{t}'_{ij}(q) - \frac{r_i(\underline{t}_{ij}(q))-K}{\eta_i^s}\delta_i(\underline{t}_{ij}(q);q)\underline{t}'_{ij}(q)\right] \\
& - \sum_{i=1}^{2}\sum_{j=1}^{N}\left[\frac{\tilde{r}_i(\bar{\tau}_{ij}(q))}{\eta_i^l}\tilde{\delta}_i(\bar{\tau}_{ij}(q);q)\bar{\tau}'_{ij}(q) - \frac{\tilde{r}_i(\underline{\tau}_{ij}(q))}{\eta_i^l}\tilde{\delta}_i(\underline{\tau}_{ij}(q);q)\underline{\tau}'_{ij}(q)\right],
\end{aligned}
$$

where the last two summations vanish because each of the summands is zero (by definition of the trigger functions and $\delta_i, \tilde{\delta}_i$) regardless of whether one uses the left or the right derivatives. Then because the trigger functions are continuous, F is continuously differentiable with

$$
F'(q) = \sum_{i=1}^{2}\sum_{j=1}^{N}\int_{\underline{t}_{ij}(q)}^{\bar{t}_{ij}(q)} \frac{(r_i(t)-K)^2}{\eta_i^s}dt + \sum_{i=1}^{2}\sum_{j=1}^{N}\int_{\underline{\tau}_{ij}(q)}^{\bar{\tau}_{ij}(q)} \frac{(\tilde{r}_i(t))^2}{\eta_i^l}dt.
$$

Moreover, $F'(q) > 0$ because $0 \leq \underline{t}_{i1}(q) < \bar{t}_{i1}(q) \leq T$ for some i by assumption (5). Therefore, $F(q)$ is strictly increasing on (0,1). ∎

Proof of Theorem 1. Note that the hospice manager's problem (P) is equivalent to the optimal control problem (40)-(46) presented in Appendix B, where $\alpha_i(t) = \dot{z}_i(t)$ and $\theta_i(t) = \dot{\Theta}_i(t)$ for all i, t. Similarly, the dual optimal control problem (60)-(65) is equivalent to the dual problem (D), introduced in Appendix B, with $\dot{p}(t) = 0$ and $p(t) = (K(1-q), q)$ for all t. Rockafellar (1970) provides a duality relationship between (40)-(46) and (60)-(65), whereby the two formulations have the same optimal objective. Moreover, by Theorem 5 of Rockafellar (1970), the optimal solutions to (40)-(46) and (60)-(65), must satisfy

$$(\dot{p}(t), p(t)) \in \partial L(t, (z(t), \zeta(t)), (\dot{z}(t), \dot{\zeta}(t))) \tag{1}$$

for $t \in [0, T]$. Also note by Proposition 8.12 of Rockafellar and Wets (1997) that for any proper convex function f, its subgradient set $\partial f(\bar{x})$ at $\bar{x}$ is given by

$$\partial f(\bar{x}) = \{u : f(x) \geq f(\bar{x}) + \langle u, x - \bar{x} \rangle \text{ for all } x\}.$$

Namely, for $v \in \partial f(\bar{x})$, we must have that $f(x) \geq f(\bar{x}) + \langle v, x \rangle - \langle v, \bar{x} \rangle$ for all x. Rearranging the terms gives

$$\langle v, \bar{x} \rangle - f(\bar{x}) \geq \langle v, x \rangle - f(x) \text{ for all } x,$$

which holds with equality for $x = \bar{x}$. Therefore,

$$\langle v, \bar{x} \rangle - f(\bar{x}) = \sup_x \{\langle v, x \rangle - f(x)\} = f^*(v). \tag{2}$$

Hence, we conclude that $v \in \partial f(\bar{x})$ if and only if $\bar{x}$ is an element of the set $\arg\max_x \{\langle v, x \rangle - f(x)\}$ in defining $f^*(v)$, c.f., (2). Using this observation, (1) holds if and only if, for $t \in [0, T]$,

$$(z(t), \Theta(t), \zeta(t), \dot{z}(t), \dot{\Theta}(t), \dot{\zeta}(t)) \in \arg\max_{x,y} \{(\dot{p}(t), p(t)) \cdot (x, y) - L(t, x, y)\}. \tag{3}$$

By Proposition 6, (3) is equivalent to having

$$\dot{z}_i(t) = \frac{[p_i^z + r_i(t) p_i^\zeta + v_i(t) - c_i(t)]^+}{\eta_i^s} \text{ for } i = 1, 2. \tag{4}$$

$$\dot{\Theta}_i(t) = \frac{[p_i^\Theta - \tilde{r}_i(t) p_i^\Theta + \tilde{c}_i(t) - \tilde{v}_i(t)]^+}{\eta_i^l} \text{ for } i = 1, 2. \tag{5}$$

$$\dot{\zeta}_i(t) = r_i(t) \dot{z}_i(t) - \tilde{r}_i(t) \dot{\Theta}_i(t), \text{ for } i = 1, 2.$$

Also, note by equivalence of the formulations (60)-(65) and (D) that

$$p_i^z = K(1-q), p_i^\zeta = q \quad \text{and} \quad p_i^\Theta = 0 \text{ for } i = 1, 2.$$

Similarly, by the equivalence of (40)-(46) and (P) we have that

$$\alpha(t) = \dot{z}(t) \text{ and } \theta(t) = \dot{\Theta}(t) \text{ for } t \in [0, T].$$

Therefore, (4)-(5) give the hospice manager's optimal recruiting and live-discharge rates as stated in the theorem, where q^* is the optimal solution to the dual problem, characterized in Proposition 7. ∎

Proof of Proposition 4. Let $\alpha(\cdot)$ and $\theta(\cdot)$ denote a feasible nonstationary policy. Then let

$$\bar{\alpha}_i = \frac{1}{T}\int_0^T \alpha_i(s)ds \text{ and } \bar{\theta}_i = \frac{1}{T}\int_0^T \theta_i(s)ds,$$

and observe that for $i = 1, 2$,

$$\int_0^T \frac{1}{T} s_i(\alpha_i(t))dt > s_i\left(\int_0^T \frac{1}{T}\alpha_i(t)dt\right) = s_i(\bar{\alpha}_i) \tag{6}$$

$$\int_0^T \frac{1}{T} g_i(\theta_i(t))dt > g_i\left(\int_0^T \frac{1}{T}\theta_i(t)dt\right) = g_i(\bar{\theta}_i) \tag{7}$$

by Jensen's inequality and strict convexity of the quadratic recruiting and live-discharging cost functions $s_i(\cdot)$ and $g_i(\cdot)$. Multiplying both sides of (6)-(7) by T gives

$$\int_0^T s_i(\alpha_i(t))dt > Ts_i(\bar{\alpha}_i) \text{ and } \int_0^T g_i(\theta_i(t))dt > Tg_i(\bar{\theta}_i)$$

so that the recruiting and live-discharge costs are strictly larger for the nonstationary policy $\alpha(\cdot)$ and $\theta(\cdot)$ than its stationary counterpart $\bar{\alpha}$ and $\bar{\theta}$, while all other costs and the revenues are the same for the two policies. Thus, switching over to the stationary policy $(\bar{\alpha}, \bar{\theta})$ strictly improves the hospice's profit. ∎

Proof of Lemma 1. For notational simplicity, let

$$a = \sum_{i=1}^{2}(rm_i - K)\left[\lambda_i + \frac{(r-c_i)m_i}{\eta_i^s}\right] \text{ and } b = \sum_{i=1}^{2}\frac{(rm_i - K)^2}{\eta_i^s}.$$

(i) Note that $\pi_1 < c_2/r$ is equivalent to

$$br + \sum_{i=1}^{2}\frac{(r\tilde{m}_i)^2}{\eta_i^l}c_i - ra < c_2 b + c_2\sum_{i=1}^{2}\frac{(r\tilde{m}_i)^2}{\eta_i^l}. \tag{8}$$

Subtracting $c_2(r\tilde{m}_2)^2/\eta_2^l$ from both sides of (8) shows that (8) is equivalent to

$$br + \frac{(r\tilde{m}_1)^2}{\eta_1^l}c_1 - ra < c_2 b + c_2\frac{(r\tilde{m}_1)^2}{\eta_1^l}, \tag{9}$$

which, in turn, is equivalent to $\pi_2 < c_2/r$.

(ii) Because $\pi_1 < c_2/r$ is equivalent to (9), it follows from $\pi_1 < c_2/r$ and $c_1 > c_2$ that

$$br + \frac{(r\tilde{m}_1)^2}{\eta_1^l}c_1 - ra < c_2 b + \frac{c_2(r\tilde{m}_1)^2}{\eta_1^l} < c_1 b + c_1\frac{(r\tilde{m}_1)^2}{\eta_1^l},$$

which implies $\pi_3 < c_1/r$.

(iii) Note that $\pi_2 < c_1/r$ is equivalent to

$$br + \frac{(r\tilde{m}_1)^2}{\eta_1^l}c_1 - ra < bc_1 + \frac{c_1(r\tilde{m}_1)^2}{\eta_1^l},$$

which clearly is equivalent to $\pi_3 < c_1/r$. ■

Proof of Proposition 5. We can make the hospice manager's optimization problem convex and put it in the form of Boyd and Vandenberghe (2004) as follows:

$$\begin{aligned}
\min -\xi + \sum_{i=1}^{2} c_i m_i \alpha_i - \sum_{i=1}^{2} c_i \tilde{m}_i \theta_i + \sum_{i=1}^{2} \frac{1}{2}\eta_i^s \alpha_i^2 \quad &+ \quad \sum_{i=1}^{2} \frac{1}{2}\eta_i^l \theta_i^2 \\
\text{subject to } \xi - K\sum_{i=1}^{2}(\lambda_i + \alpha_i) \quad &\leq \quad 0, \\
\xi - r\sum_{i=1}^{2}[m_i(\lambda_i + \alpha_i) - \tilde{m}_i\theta_i] \quad &\leq \quad 0, \\
-\alpha_i \quad &\leq \quad 0, i = 1, 2. \\
-\theta_i \quad &\leq \quad 0, i = 1, 2.
\end{aligned}$$

Letting μ_i, ν_i^s, ν_i^l for $i = 1, 2$ denote the Lagrange multipliers, we write the KKT conditions as follows (see p. 243 of Boyd and Vandenberghe, 2004). First consider setting the gradient of the Lagrangian to zero, which gives

$$\begin{aligned}
&-1 + \mu_1 + \mu_2 = 0, \\
&c_i m_i + \eta_i^s \alpha_i - K\mu_1 - r m_i \mu_2 - \nu_i^s = 0, \qquad i = 1, 2 \\
&-c_i \tilde{m}_i + \eta_i^l \theta_i + r\tilde{m}_i \mu_2 - \nu_i^l = 0, \qquad i = 1, 2.
\end{aligned}$$

That is,

$$\alpha_i = \frac{K\mu_1 + r m_i \mu_2 + \nu_i^s - c_i m_i}{\eta_i^s}, \tag{10}$$

$$\theta_i = \frac{-r\tilde{m}_i \mu_2 + \nu_i^l + c_i \tilde{m}_i}{\eta_i^l}, \tag{11}$$

$$\mu_1 + \mu_2 = 1. \tag{12}$$

Also, the feasibility and complementary slackness conditions give the following:

$$\xi - K\sum_{i=1}^{2}(\lambda_i + \alpha_i) \leq 0, \tag{13}$$

$$\xi - r\sum_{i=1}^{2}[m_i(\lambda_i + \alpha_i) - \tilde{m}_i\theta_i] \leq 0, \tag{14}$$

$$\alpha_i \geq 0, \tag{15}$$

$$\theta_i \geq 0, \tag{16}$$

$$\mu_1\left[\xi - K\sum_{i=1}^{2}(\alpha_i + \lambda_i)\right] = 0, \tag{17}$$

$$\mu_2\left[\xi - r\sum_{i=1}^{2}[m_i(\alpha_i + \lambda_i) - \tilde{m}_i\theta_i]\right] = 0, \tag{18}$$

$$\nu_i^s\alpha_i = 0, \quad i = 1, 2 \tag{19}$$

$$\nu_i^l\theta_i = 0, \quad i = 1, 2 \tag{20}$$

$$\mu_i \geq 0, \quad i = 1, 2 \tag{21}$$

$$\nu_i^j \geq 0 \quad i = 1, 2, j = s, l. \tag{22}$$

Then equations (10)-(22) will pin down the optimal solution.

Recall that $K > c_i m_i$ for $i = 1, 2$ by assumption. Then it is easy to see that $\alpha_i > 0$ for $i = 1, 2$. Otherwise, by (10), $\alpha_i = 0$ implies

$$\nu_i^s = m_i[c_i - (\mu_2 r + (1 - \mu_2)K/m_1] < 0,$$

which contradicts (22). Therefore, in what follows we assume $\alpha_i > 0$ for $i = 1, 2$, which, in turn, implies by (19) that

$$\nu_i^s = 0 \text{ for } i = 1, 2. \tag{23}$$

Despite this simplification, we still need to consider several cases. First, note that at least one of the inequalities (13) and (14) will bind. Therefore, we have the following three main cases to consider:

- Case 1: Constraint (13) does not bind.
- Case 2: Constraint (14) does not bind.
- Case 3: Both (13) and (14) bind.

Consider Case 1: Because the cap constraint, c.f. (13), does not bind, one would expect that the hospice manager recruits patients and does not live-discharge any, i.e. $\theta_i = 0$ for $i = 1, 2$. Indeed, to see this note that $\mu_1 = 0$ and $\mu_2 = 1$ by (17) and (12). Also, if $\theta_i > 0$, then $\nu_i^l = 0$ by (20). Then by (11), $\theta_i = \tilde{m}_i(c_i - r)/\eta_i^l < 0$, which contradicts (16). Therefore, in case 1, we must have $\theta_i = 0$ for $i = 1, 2$. More specifically, because $\mu_1 = 0$ and $\mu_2 = 1$, we conclude from (10) and (23) that

$$\alpha_i = \frac{m_i(r - c_i)}{\eta_i^s} \qquad \text{and} \qquad \theta_i = 0 \text{ for } i = 1, 2. \tag{24}$$

To conclude the part 1 of the proof, we note that case 1 arises if and only if (13) does not bind, but (14) does:

$$\xi < K \sum_{i=1}^{2} (\lambda_i + \alpha_i) \qquad \text{and} \qquad \xi = r \sum_{i=1}^{2} \left[m_i(\lambda_i + \alpha_i) - \tilde{m}_i \theta_i \right].$$

That is,

$$r \sum_{i=1}^{2} \left[m_i(\lambda_i + \alpha_i) - \tilde{m}_i \theta_i \right] < K \sum_{i=1}^{2} (\lambda_i + \alpha_i).$$

Substituting (24) and rearranging terms gives the following equivalent condition:

$$\sum_{i=1}^{2} (rm_i - K) \left[\lambda_i + \frac{m_i(r - c_i)}{\eta_i^s} \right] < 0 \tag{25}$$

which along with (24) prove part 1 of Proposition 5.

Next, consider case 2: constraint (14) does not bind. That is, the cap constraint binds and the potential revenue constraint does not. In this case, one would expect the hospice manager to both recruit patients (to relax the cap constraint) and to live-discharge patients to reduce costs. In other words, we expect to see $\theta_i > 0$ for $i = 1, 2$. Indeed, to see this note that by (18) and (12), $\mu_1 = 1, \mu_2 = 0$. Then it follows from (11) that $\theta_i \geq c_i \tilde{m}_i / \eta_i^l > 0$ for $i = 1, 2$. Therefore, we must have $\theta_i > 0$ for $i = 1, 2$ in case 2.

To be more specific, $\theta_i > 0$ implies $\nu_i^l = 0$ by (20). Then, it follows from (10)-(11) and (23) that

$$\alpha_i = \frac{K - c_i m_i}{\eta_i^s} \qquad \text{and} \qquad \theta_i = \frac{c_i \tilde{m}_i}{\eta_i^l} \qquad \text{for } i = 1, 2. \tag{26}$$

Moreover, observe that case 2 arises if and only if

$$\xi = K \sum_{i=1}^{2} (\lambda_i + \alpha_i) \qquad \text{and} \qquad \xi < r \sum_{i=1}^{2} \left[m_i(\lambda_i + \alpha_i) - \tilde{m}_i \theta_i \right].$$

That is,

$$K \sum_{i=1}^{2} (\lambda_i + \alpha_i) < r \sum_{i=1}^{2} \left[m_i(\lambda_i + \alpha_i) - \tilde{m}_i \theta_i \right].$$

Substituting (26) into this and rearranging terms shows that case 2 arises if and only if

$$\sum_{i=1}^{2} \frac{r(\tilde{m}_i)^2 c_i}{\eta_i^l} + \sum_{i=1}^{2} \frac{(rm_i - K)}{\eta_i^s} < \sum_{i=1}^{2} (rm_i - K) \left[\lambda_i + \frac{m_i(r - c_i)}{\eta_i^s} \right], \tag{27}$$

which along with (26) proves part 2 of Proposition 5.

Finally, consider case 3, where both (13) and (14) bind. As a first step, we rule out the case $\theta_1 = 0$ and $\theta_2 > 0$. Suppose that this is possible. Then by (20), $\theta_2 > 0$ implies $\nu_2^l = 0$, and by (11) we write $\theta_2 = \tilde{m}_2(c_2 - \mu_2 r)/\eta_2^l$, which in turn implies $\mu_2 < c_2/r$ because $\theta_2 > 0$. At the same time, $\theta_1 = 0$ implies by (11) that $\mu_2 = \nu_2^l/(r\tilde{m}_1) + c_1/r \geq c_1/r$. Thus, we conclude both $\mu_2 \geq c_1/r$ and $\mu_2 < c_1/r$, which is a contradiction because $c_1 > c_2$. Next, we consider the following three subcases, each of which may arise:

Case 3a: $\theta_i > 0$ for $i = 1, 2$.

Case 3b: $\theta_1 > 0$ and $\theta_2 = 0$.

Case 3c: $\theta_i = 0$ for $i = 1, 2$.

Consider Case 3a: $\theta_i > 0$ implies $\nu_i^l = 0$ for $i = 1, 2$ by (20). Then it follows from (10)-(11) and (23) that

$$\alpha_i = \frac{K(1-\mu_2) + rm_i\mu_2 - c_i m_i}{\eta_i^s} \qquad \text{and} \qquad \theta_i = \frac{\tilde{m}_i(c_i - r\mu_2)}{\eta_i^l} \text{ for } i = 1, 2. \tag{28}$$

Because we must have $\theta_i > 0$, (28) requires that

$$\mu_2 < \frac{c_2}{r} < \frac{c_1}{r}. \tag{29}$$

Moreover, because both (13) and (14) bind, we must have that

$$K\sum_{i=1}^{2}(\alpha_i + \lambda_i) = r\sum_{i=1}^{2}[m_i(\lambda_i + \alpha_i) - \tilde{m}_i\theta_i]. \tag{30}$$

Substituting $\mu_1 = 1 - \mu_2$ and (27) into (30) and rearranging terms give

$$\mu_2 = \pi_1. \tag{31}$$

Consider Case 3b: $\theta_1 > 0$ implies that $\nu_1^l = 0$. Moreover, $\theta_2 = 0$ implies by (11) that $\nu_2^l = \tilde{m}_2(r\mu_2 - c_2) \geq 0$, which requires $\mu_2 \geq c_2/r$. Also, $\theta_1 > 0$ requires $\mu_2 < c_1/r$ by (10). So we must have

$$\frac{c_2}{r} \leq \mu_2 < \frac{c_1}{r}. \tag{32}$$

Also, it follows from (10)-(11) and (23) that

$$\alpha_i = \frac{K(1-\mu_2) + rm_i\mu_2 - c_i m_i}{\eta_i^s} \qquad \text{for } i = 1, 2 \text{ and} \qquad \theta_1 = \frac{\tilde{m}_1(c_1 - r\mu_2)}{\eta_1^l}. \tag{33}$$

Because both (13) and (14) bind, (30) must hold. Then using (33) and the fact that $\theta_2 = 0$, we find that

$$\mu_2 = \pi_2. \tag{34}$$

Finally, consider Case 3c: $\theta_2 = 0$ for $i = 1, 2$. Note that $\theta_i = 0$ implies by (11) that $\nu_i^l = \tilde{m}_i[r\mu_2 - c_i]$, which, in turn, implies $\mu_2 \geq c_i/r$. That is, we must have

$$\mu_2 \geq \frac{c_1}{r} > \frac{c_2}{r}. \tag{35}$$

Moreover, it follows from (10) and (23) that

$$\alpha_i = \frac{K(1-\mu_2) + rm_i\mu_2 - c_i m_i}{\eta_i^s} \qquad \text{for } i = 1, 2. \tag{36}$$

Because both (13) and (14) bind, using (35) and the fact that $\theta_i = 0$ for $i = 1, 2$, one can show that

$$\mu_2 = \pi_3. \tag{37}$$

To conclude the proof, note by Lemma 1 that the conditions (29) and (31), (32) and (34), and (35) and (37) partition the remaining subcases of case 3 above (over the parameter space of the problem) as in Part 3 of Proposition 5. Combining this partition and characterizations of recruiting and live-discharge rates in (28), (33), and (36) proves Part 3 of Proposition 5. ■

B Duality Analysis

In this appendix, we derive the dual formulation and some auxiliary results. To facilitate the statement of the dual formulation, let

$$\beta_1 = K(\lambda_1 + \lambda_2)T \tag{38}$$

$$\beta_2 = \sum_{i=1}^{2} \int_0^T \lambda_i r_i(s)ds + R(0) \tag{39}$$

It will be shown below that the dual formulation (D) of (P) can be stated as follows: Choose q so as to

$$\begin{aligned} \min \beta_1(1-q) + \beta_2 q + \textstyle\sum_{i=1}^2 \int_0^T \frac{1}{2\eta_i^s} \left([K(1-q) + r_i(t)q + v_i(t) - c_i(t)]^+\right)^2 dt \\ + \textstyle\sum_{i=1}^2 \int_0^T \frac{1}{2\eta_i^l} \left([-\tilde{r}_i(t)q - \tilde{v}_i(t) + \tilde{c}_i(t)]^+\right)^2 dt \\ \text{subject to} \qquad 0 \leq q \leq 1. \end{aligned} \tag{D}$$

This appendix proves this statement and (in Proposition 7) provides the optimal solution to this dual formulation.

For $i = 1, 2$ and $t \in [0, T]$, let

$$z_i(t) = \int_0^t \alpha_i(s)ds \qquad \text{and} \qquad \Theta_i(t) = \int_0^t \theta_i(s)ds$$

(in particular, $\dot{z}_i(s) = \alpha_i(s)$ and $\dot{\Theta}_i(s) = \theta_i(s)$). That is, $z_i(t)$ and $\Theta_i(t)$ denote the cumulative number of recruited and live-discharged, respectively, patients by time t. Observe that the hospice manager's problem (P) can be written as follows: Choose $\dot{z}(\cdot), \dot{\Theta}(\cdot)$ and $\dot{\zeta}(\cdot)$ so as to

$$\min\{-\min(\beta_1 + Kz_1(T) + Kz_2(T), \beta_2 + \zeta_1(T) + \zeta_2(T))\} + \int_0^T \sum_{i=1}^{2}[(c_i(t) - v_i(t))\dot{z}_i(t) + s_i(\dot{z}_i(t))]dt$$
$$- \int_0^T \sum_{i=1}^{2}[(\tilde{c}_i(t) - \tilde{v}_i(t))\dot{\Theta}_i(t) - g_i(\dot{\Theta}_i(t))]dt \tag{40}$$

subject to

$$z_i(t) = z_i(0) + \int_0^t \dot{z}_i(s)ds, z_i(0) = 0, \tag{41}$$
$$\zeta_i(t) = \zeta_i(0) + \int_0^t \dot{\zeta}_i(s)ds, \zeta_i(0) = 0, \tag{42}$$
$$\Theta_i(t) = \Theta_i(0) + \int_0^t \dot{\Theta}_i(s)ds, \Theta_i(0) = 0, \tag{43}$$
$$\dot{z}_i(s) \geq 0, \tag{44}$$
$$\dot{\Theta}_i(s) \geq 0, \tag{45}$$
$$\dot{\zeta}_i(s) = r_i(s)\dot{z}_i(s) - \tilde{r}_i(s)\dot{\Theta}_i(s). \tag{46}$$

The cumulative revenue accrued until time t from patients of type i is $\zeta_i(t)$ and $\dot{\zeta}_i(t)$ is the revenue rate at time t from patients of type i. To put this in the framework of Rockafellar (1970), define

$$L(t, x, y) = \sum_{i=1}^{2}[(c_i(t) - v_i(t))y_i^z + s_i(y_i^z) + \chi_{\{y_i^z \geq 0\}}] \tag{47}$$
$$+ \sum_{i=1}^{2}[-(\tilde{c}_i(t) - \tilde{v}_i(t))y_i^\Theta + g_i(y_i^\Theta) + \chi_{\{y_i^\Theta \geq 0\}}] + \sum_{i=1}^{2}\chi_{\{y_i^\zeta = r_i(t)y_i^z - \tilde{r}_i(t)y_i^\Theta\}}], \tag{48}$$
$$l(x, y) = \chi_{\{x^z=0, x^\Theta=0, x^\zeta=0\}} - \min\{\beta_1 + Ky_1^z + Ky_2^z, \beta_2 + y_1^\zeta + y_2^\zeta\} \tag{49}$$

for $x_i = (x_i^z, x_i^\Theta, x_i^\zeta) \in \Re^3$, $x = (x_1, x_2) \in \Re^6$, $y_i = (y_i^z, y_i^\Theta, y_i^\zeta) \in \Re^3$, $y = (y_1, y_2) \in \Re^6$, and $x^j = (x_1^j, x_2^j)$, $y^j = (y_1^j, x_2^j) \in \Re^2$ for $j = z, \Theta, \zeta$, and $\chi_F(\cdot)$ is an "indicator" function taking values zero or infinity. Namely,

$$\chi_F(a) = \begin{cases} \infty & \text{if } a \notin F, \\ 0 & \text{otherwise.} \end{cases}$$

Notice $L(t,x,y)$ is independent of x because the profit function is independent of the cumulative number of recruited patients and cumulative revenue *within* the year; the cumulative number of patients and cumulative revenue is relevant only at time T which are represented by y in $l(x,y)$. Then the hospice manager's problem can be written as follows: Choose the functions $\dot{z}(\cdot), \dot{\Theta}(\cdot), \dot{\zeta}(\cdot)$ so as to

$$\min l((z(0),\Theta(0),\zeta(0)),(z(T),\Theta(T),\zeta(T))) + \int_0^T L(t,(z(t),\Theta(t),\zeta(t)),(\dot{z}(t),\dot{\Theta}(t),\dot{\zeta}(t)))dt. \quad (50)$$

Following Rockafellar (1970) to derive the dual problem,[1] define

$$\begin{aligned} m(d(0),d(T)) &= l^*(d(0),-d(T)), && (51)\\ M(t,p,s) &= L^*(t,s,p), && (52) \end{aligned}$$

where l^* and L^* are convex conjugates of l and L, respectively, m is the terminal function dual to l, M is the Lagrangian function dual to L, and $d(t) = (d^z(t), d^\zeta(t)) \in \Re^6$. The dual problem can then be stated as follows: Choose $p(\cdot)$ and $\dot{p}(\cdot)$ so as to

$$\min m(p(0),p(T)) + \int_0^T M(t,p(t),\dot{p}(t))dt. \quad (53)$$

The next step is to characterize m, M which is done in the next proposition.

Proposition 6 *We have that*

$$\begin{aligned} m(d^z(0),d^\Theta(0),d^\zeta(0),d^z(T),d^\Theta(T),d^\zeta(T)) &= \beta_1 d_1^z(T)/K + \beta_2 d_1^\zeta(T) + \chi_{\{d_1^j(T)=d_2^j(T), j=z,\zeta\}} \\ &\quad + \chi_{\{d_1^z(T)+Kd_2^\zeta(T)=1\}} + \chi_{\{d_i^j(T)\geq 0, i=1,2, j=z,\zeta\}} + \chi_{\{d^\Theta(T)=0\}}, \quad (54)\\ M(t,p,s) &= \sum_{i=1}^2 \frac{1}{2\eta_i^s}([p_i^z + r_i(t)p_i^\zeta + v_i(t) - c_i(t)]^+)^2 + \chi_{\{s\equiv 0\}} \\ &\quad + \sum_{i=1}^2 \frac{1}{2\eta_i^l}([p_i^\Theta - \tilde{r}_i(t)p_i^\zeta - \tilde{v}_i(t) + \tilde{c}_i(t)]^+)^2. \quad (55) \end{aligned}$$

Moreover,

$$\arg\max_{x,y}\{(s,p)\cdot(x,y) - L(t,x,y)\} = \left\{ (x,y) : \begin{array}{l} y_i^z = \frac{[p_i^z + r_i(t)p_i^\zeta + v_i(t) - c_i(t)]^+}{\eta_i^s}, \\ y_i^\Theta = \frac{[p_i^\Theta - \tilde{r}_i(t)p_i^\Theta - \tilde{v}_i(t) + \tilde{c}_i(t)]^+}{\eta_i^l}, \text{ and} \\ y_i^\zeta = r_i(t)y_i^z - \tilde{r}_i(t)y_i^\Theta \end{array} \right\}. \quad (56)$$

[1]Here we are following Rockafellar's (1970) notation as closely as possible to facilitate the use of his results. For example, the swapping of the order of arguments in M and L^* in (52) matches his equation (5.5) on p. 190.

Proof. Note that

$$\begin{aligned}
m(d^z(0), d^\Theta(0), d^\zeta(0), d^z(T), d^\Theta(T), d^\zeta(T)) &= l^*(d^z(0), d^\Theta(0), d^\zeta(0), -d^z(T), -d^\Theta(T), -d^\zeta(T)) \\
&= \sup_c \{c(0)d(0) - c(T)d(T) - \chi_{\{c(0)=0\}} \\
&\quad + \min\{\beta_1 + Kc_1^z(T) + Kc_2^z(T), \beta_2 + c_1^\zeta(T) + c_2^\zeta(T)\}\} \\
&= \sup_{c(T)} \{-c^z(T)d^z(T) - c^\zeta(T)d^\zeta(T) - c^\Theta(T)d^\Theta(T) \\
&\quad + \min\{\beta_1 + Kc_1^z(T) + Kc_2^z(T), \beta_2 + c_1^\zeta(T) + c_2^\zeta(T)\}\} \\
&= \chi_{\{d^\Theta(T)=0\}} + \sup_{c(T)} \{-c^z(T)d^z(T) - c^\zeta(T)d^\zeta(T) \\
&\quad + \min\{\beta_1 + Kc_1^z(T) + Kc_2^z(T), \beta_2 + c_1^\zeta(T) + c_2^\zeta(T)\}\}.
\end{aligned}$$

The optimization problem on the right-hand side is equivalent to the following linear program:

$$\max_{c(T)} \left\{\xi - c_1^z(T)d_1^z(T) - c_2^z(T)d_2^z(T) - c_1^\zeta(T)d_1^\zeta(T) - c_2^\zeta(T)d_2^\zeta(T)\right\}$$

subject to

$$\begin{aligned}
\xi &\leq \beta_1 + Kc_1^z(T) + Kc_2^z(T), \\
\xi &\leq \beta_2 + c_1^\zeta(T) + c_2^\zeta(T).
\end{aligned}$$

The dual linear program is given by

$$\min \beta_1 y_1 + \beta_2 y_2$$

subject to

$$\begin{bmatrix} -K & 0 \\ -K & 0 \\ 0 & -1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -d_1^z(T) \\ -d_2^z(T) \\ -d_1^\zeta(T) \\ -d_2^\zeta(T) \\ 1 \end{pmatrix},$$

$$y \geq 0,$$

whose constraints are equivalent to the following:

$$\begin{aligned}
&d_1^j(T) = d_2^j(T) \text{ for } j = z, \zeta, \\
&d_1^z(T)/K + d_1^\zeta(T) = 1, \\
&d_i^j(T) \geq 0 \text{ for } i = 1, 2,\ j = z, \zeta.
\end{aligned}$$

Moreover, its objective is given by $\beta_1 d_1^z(T)/K + \beta_2 d_1^\zeta(T)$, from which (54) follows.

Similarly,

$$M(t,p,s) = L^*(t,s,p),$$

where

$$\begin{aligned} L^*(t,s,p) &= \sup_{x,y}\{(s,p)\cdot(x,y) - L(t,x,y)\} \\ &= \sup_{x_z,x_\Theta,x_\zeta,y_z,y_\Theta,y_\zeta} \Big\{ x_1^z s_1^z + x_2^z s_2^z + x_1^\zeta s_1^\zeta + x_2^\zeta s_2^\zeta + y_1^z p_1^z + y_2^z p_2^z + y_1^\zeta p_1^\zeta + y_2^\zeta p_2^\zeta \\ &\quad - \sum_{i=1}^2 [(c_i(t) - v_i(t))y_i^z + \frac{1}{2}\eta_i^s (y_i^z)^2] \\ &\quad + \sum_{i=1}^2 [(\tilde{c}_i(t) - \tilde{v}_i(t))y_i^\Theta - \frac{1}{2}\eta_i^l (y_i^\Theta)^2] : y_i^z \geq 0, y_i^\Theta \geq 0, y_i^\zeta = r_i(t) y_i^z - \tilde{r}_i(t) y_i^\Theta \Big\}. \end{aligned}$$

It is clear that we must have $s \equiv 0$ (and that x^z, x^ζ can take any value). Then

$$\begin{aligned} L^*(t,s,p) &= \chi_{\{s=0\}} + \sup_{y^z,y^\Theta} \Big\{ y_1^z p_1^z + y_2^z p_2^z + y_1^\Theta p_1^\Theta + y_2^\Theta p_2^\Theta + (r_1(t) y_1^z - \tilde{r}_1(t) y_1^\Theta) p_1^\zeta \\ &\quad + (r_2(t) y_2^z - \tilde{r}_2(t) y_2^\Theta) p_2^\zeta - \sum_{i=1}^2 [(c_i(t) - v_i(t))y_i^z + \frac{1}{2}\eta_i^s (y_i^z)^2] \\ &\quad + \sum_{i=1}^2 [(\tilde{c}_i(t) - \tilde{v}_i(t))y_i^\Theta - \frac{1}{2}\eta_i^l (y_i^\Theta)^2] : y_i^z, y_i^\Theta \geq 0 \Big\}, \end{aligned}$$

where we substituted $y_i^\zeta = r_i(t) y_i^z - \tilde{r}_i(t) y_i^\Theta$. Notice that the optimization problem on the right-hand side decomposes so that

$$\begin{aligned} L^*(t,s,p) &= \chi_{\{s=0\}} + \sum_{i=1}^2 \sup_{y_i^z} \Big\{ y_i^z (p_i^z + r_i(t) p_i^\zeta - c_i(t) + v_i(t)) - \frac{1}{2}\eta_i^s (y_i^z)^2 : y_i^z \geq 0 \Big\} \\ &\quad + \sum_{i=1}^2 \sup_{y_i^\Theta} \Big\{ y_i^\Theta (p_i^\Theta - \tilde{r}_i(t) p_i^\zeta + \tilde{c}_i(t) - \tilde{v}_i(t)) - \frac{1}{2}\eta_i^l (y_i^\Theta)^2 : y_i^\Theta \geq 0 \Big\}. \end{aligned} \tag{57}$$

Then the first order conditions give

$$y_i^z = \frac{[p_i^z + r_i(t) p_i^\zeta + v_i(t) - c_i(t)]^+}{\eta_i^s}, \tag{58}$$

$$y_i^\Theta = \frac{[p_i^\Theta - \tilde{r}_i(t) p_i^\zeta - \tilde{v}_i(t) + \tilde{c}_i(t)]^+}{\eta_i^l}. \tag{59}$$

Substituting these into (57) and using the definition $M(t,p,s) = L^*(t,s,p)$ gives (55). Moreover, (56) follows from (58)-(59), and the fact that $y_i^\zeta = r_i(t) y_i^z - \tilde{r}_i(t) y_i^\Theta$ and that x can take any value because $s \equiv 0$. ∎

Then combining Proposition 6 with (53) gives the dual formulation: Choose $p(\cdot), \dot{p}(\cdot)$ so as to

$$\min \beta_1 p_1^z(T)/K + \beta_2 p_1^\zeta(T) + \sum_{i=1}^{2} \int_0^T \frac{1}{2\eta_i^s}([p_i^z(t) + r_i(t)p_i^\zeta(t) + v_i(t) - c_i(t)]^+)^2 dt$$
$$+ \sum_{i=1}^{2} \int_0^T \frac{1}{2\eta_i^l}([p_i^\Theta(t) - \tilde{r}_i(t)p_i^\zeta(t) - \tilde{v}_i(t) + \tilde{c}_i(t)]^+)^2 dt \tag{60}$$

subject to

$$p_1^j(T) = p_2^j(T) \text{ for } j = z, \zeta, \tag{61}$$

$$p_1^z(T) + Kp_1^\zeta(T) = K, \tag{62}$$

$$p_i^j(T) \geq 0 \text{ for } i = 1, 2,\ j = z, \zeta, \tag{63}$$

$$p_i^\Theta(T) = 0 \text{ for } i = 1, 2. \tag{64}$$

$$\dot{p}_i(t) = 0 \text{ for } i = 1, 2 \text{ and } t \geq 0. \tag{65}$$

Note that $p(t) = p(T)$ for all $t \in [0, T]$ because $\dot{p}(t) = 0$. Then letting $q = p_1^\zeta(T)$, using the constraint that $p_1^z(T) + Kp_1^\zeta(T) = K$, the hospice manager's problem reduces to

$$\min \beta_1(1-q) + \beta_2 q + \sum_{i=1}^{2} \int_0^T \frac{1}{2\eta_i^s} \left([K(1-q) + r_i(t)q + v_i(t) - c_i(t)]^+\right)^2 dt$$
$$+ \sum_{i=1}^{2} \int_0^T \frac{1}{2\eta_i^l} \left([-\tilde{r}_i(t)q - \tilde{v}_i(t) + \tilde{c}_i(t)]^+\right)^2 dt$$

subject to $0 \leq q \leq 1$.

Proposition 7 *The optimal solution q^* of the dual formulation is given as follows:*

$$q^* = \begin{cases} 0 & \text{if } F(0) \geq 0, \\ F^{-1}(0) & \text{if } F(0) < 0 < F(1), \\ 1 & \text{if } F(1) \leq 0, \end{cases} \quad . \tag{66}$$

.

Proof. Letting

$$G(q) = \beta_1(1-q) + \beta_2 q + \sum_{i=1}^{2} \int_0^T \frac{1}{2\eta_i^s} \left([K(1-q) + r_i(t)q + v_i(t) - c_i(t)]^+\right)^2 dt$$
$$+ \sum_{i=1}^{2} \int_0^T \frac{1}{2\eta_i^l} \left([-\tilde{r}_i(t)q - \tilde{v}_i(t) + \tilde{c}_i(t)]^+\right)^2 dt,$$

the dual formulation can be written as follows:

$$\min G(q) \text{ subject to } 0 \leq q \leq 1. \tag{67}$$

Adopting the trigger functions constructed for the proof of Proposition 3, we can write

$$\begin{aligned} G(q) &= \beta_1(1-q) + \beta_2 q + \sum_{i=1}^{2}\sum_{j=1}^{N}\int_{\underline{t}_{ij}(q)}^{\bar{t}_{ij}(q)} \frac{1}{2\eta_i^s}\left(K(1-q) + r_i(t)q + v_i(t) - c_i(t)\right)^2 dt \\ &\quad + \sum_{i=1}^{2}\sum_{j=1}^{N}\int_{\underline{\tau}_{ij}(q)}^{\bar{\tau}_{ij}(q)} \frac{1}{2\eta_i^l}\left(-\tilde{r}_i(t)q - \tilde{v}_i(t) + \tilde{c}_i(t)\right)^2 dt, \end{aligned}$$

from which it is straightforward to show that $G' = F$. This in turn yields $G''(q) = F'(q) > 0$ for all q. Hence, G is strictly convex and the first order conditions are sufficient to pin down the unique optimal solution. Moreover, because F is strictly increasing, it follows that if $F(0) \geq 0$, then $q^* = 0$; and if $F(1) \leq 0$, then $q^* = 1$. Otherwise, we have an interior solution characterized by $F(q) = 0$, which yields $q^* = F^{-1}(0)$. ■

C Heuristic for the Simulation

This appendix describes the heuristic implemented in the simulation study. We wish to determine the total recruiting for type i (which will divided proportionally between types a and b) and the live-discharge rate for type i, b, for $i = 1, 2$. From the fluid model we can calculate $r_i^m(j)$, $c_i^m(j)$, and $v_i^m(j)$ for $i = 1, 2$, $m = a, b$, and $1 \leq j \leq 365$. Further, we define average class revenues and terminal values as $\hat{r}_i(j) = \gamma_i^a r_i^a(j) + \gamma_i^b r_i^b(j)$ and $\hat{v}_i(j) = \gamma_i^a v_i^a(j) + \gamma_i^b v_i^b(j)$ (recall that γ_i^a is the proportion of type a arrivals to class i and $\gamma_i^b = 1 - \gamma_i^a$).

Suppose that the current day is the start of day j, $1 \leq j \leq 365$. Define $R(j)$ as the potential revenue earned thus far up to day j and let $n_i(j)$ be the patients admitted thus far of type $i = 1, 2$. For $k = j, \ldots, 365$, let x_i^k be the recruiting rate for class i in period k and y_i^k the live-discharge rate for class i, b, for $i = 1, 2$. Then we estimate cumulative potential revenue for the year as

$$R = R(j) + \sum_{i=1}^{2}\sum_{k=j}^{365}[(\lambda_i + x_i^k)\hat{r}_i(k) - \sum_{i=1}^{2}\sum_{k=j}^{365}[r_i^b(k)y_i^k].$$

We estimate the cumulative cap for the year as

$$CAP = K\sum_{i=1}^{2}\left(n_i(j) + \sum_{k=j}^{365}(\lambda_i + x_i^k)\right).$$

Our mathematical program is to:

$$\begin{aligned} \max \Bigg\{ &\min(R, CAP) + \sum_{i=1}^{2}\sum_{k=j}^{365}\left((\hat{v}_i(k) - \hat{c}_i(k))(\lambda_i + x_i^k) - (v_i^b(k) - c_i^b(k))y_i^k\right) \\ &- \sum_{i=1}^{2} s_i(x_i^k) - \sum_{i=1}^{2} g_i(y_i^k)\Bigg\} \end{aligned} \tag{68}$$

subject to $x_i^k \geq 0, y_i^k \geq 0, i = 1, 2$.

This optimization can be solved analogously to that in Proposition 5 and pseudo-code for the heuristic is given below. First define:

`DayNumber` as the index of the current day.

`NatArrival[DayNumber]` as the expected revenue from natural arrivals from day `DayNumber` to the end of the year.

`patientlist[i]->length` as the number of patients of type i currently in the system.

$w_i^m(j) = r_i^m(j) - c_i^m(j) + v_i^m(j)$ and $\hat{w}_i(j) = \gamma_i^a w_i^a(j) + \gamma_i^b w_i^b(j)$

`Item[i].Eta =` η_i^s `Item[i].EtaDis =` η_i^l `HeurDenomPart1[j]=`$\sum_{k=j}^{356} (\hat{r}_i(j) - CAP)^2/\eta_i^s$

`HeurNumerPart1[j]=`$\sum_{k=j}^{356} \hat{w}_i(j)(\hat{r}_i(j) - CAP)/\eta_i^s$

`HeurDenomPart2a[j]=`$\sum_{k=j}^{356} (r_1^b(j))^2/\eta_1^l$

`HeurNumerPart2a[j]=`$\sum_{k=j}^{356} (w_1^b(j) r_1^b(j))/\eta_1^l$

`HeurDenomPart2b[j]=`$\sum_{k=j}^{356} (r_2^b(j))^2/\eta_2^l$

`HeurNumerPart2b[j]=`$\sum_{k=j}^{356} \hat{w}_2(j)(\hat{r}_2(j) - CAP)/\eta_2^s + (w_i^b(j) r_i^b(j))/\eta_i^l$

Then the discharge and recruiting rates are calculated using the following pseudo-code:

```
EstRev = YearsRevenue+NatArrival[DayNumber];
for (i=0; i<4; ++i) EstRev += r[i][DayNumber]*(patientlist[i]->length);
CapMiss = EstRev-CAP*NumAdmitTotal;
lagrange1 = -(CapMiss + HeurNumerPart1[DayNumber]+HeurNumerPart2a[DayNumber]
      +HeurNumerPart2b[DayNumber])/(HeurDenomPart1[DayNumber]
       +HeurDenomPart2a[DayNumber]+HeurDenomPart2b[DayNumber]);
lagrange2 = -(CapMiss + HeurNumerPart1[DayNumber]+HeurNumerPart2a[DayNumber])/
      (HeurDenomPart1[DayNumber]+HeurDenomPart2a[DayNumber]);
for (class=0;class<2;++class)
    if ((CapMiss+HeurNumerPart1[DayNumber] > 0)&&(lagrange1>-1))
       if (lagrange1<((c[1]-v[1])/r-1))
          recruit = (lagrange1*(r̂[class][DayNumber]-CAP) + ŵ[class][DayNumber])/(Item[class].Eta);
          discharge = (-lagrange1*r[class+2][DayNumber]- w[class+2][DayNumber])/(Item[class].EtaDis);
        else if (lagrange2<((c[0]-v[0])/r-1))
          recruit = (lagrange2*(r̂[class][DayNumber]-CAP) + ŵ[class][DayNumber])/(Item[class].Eta);
          if (class==0)
```

```
        discharge = (-lagrange2*r[class+2][DayNumber]- w[class+2][DayNumber])/
          (Item[class].EtaDis);
      else discharge =0;
    else
      lagrange3 = -(CapMiss + HeurNumerPart1[DayNumber])/(HeurDenomPart1[DayNumber]);
      recruit = (lagrange3*(r̂[class][DayNumber]-CAP) + ŵ[class][DayNumber])/(Item[class].Eta);
      discharge = 0;
    else if ((CapMiss+HeurNumerPart1[DayNumber] > 0)&&(lagrange1<=-1))
      recruit = (CAP+ŵ[class][DayNumber]-r̂[class][DayNumber])/(Item[class].Eta);
      discharge = (r[class+2][DayNumber]- w[class+2][DayNumber])/(Item[class].EtaDis);
    else
      recruit = ŵ[class][DayNumber])/(Item[class].Eta);
      discharge = 0;
RecruitingRate[i] = recruit;
DischargeRate[i] = discharge;
```

We also create an equivalent heuristic for the legacy policy. This is done by setting the values in (68) and the associated code as follows: $r_i^m(j) = rm_i^j$, $c_i^m(j) = cm_i^j$, and $v_i^m(j) = 0$ for $i = 1, 2$, $m = a, b$,

Recall that ψ yields terminal values $v_i^j = (\psi r - c_i)m_i^j$, $i = 1, 2$ and $j = a, b$. We estimate a base-case for ψ by using a steady-state version of the simulation. We assume that ψ is uniform across all patient classes and is calculated as the estimated fraction of revenue for patients that end one year that will be of use the following year. In order to calculate this, at the end of each year we calculate if there was potential reimbursement left unused at the end of the year, and if so let $\psi = 1$. Otherwise, we calculate the extra revenue received this year (including from patients who were here at the start) that went unreimbursed and let ψ equal the ratio of that unreimbursed amount to total revenue.

To be specific, terminal values are estimated as follows (note that this calculation only makes sense in a steady-state world). At the end of the year we calculate:

- `CapMiss = YearsRevenue-CAP*NumAdmitTotal` (this is the extra revenue received THIS year (including from patients who were here at the start) that went to waste or, if negative, the amount under the CAP that we ended the year with);

- `EstRev` is the projected revenue to be received by the patients left at the end of the year (using their actual death times, although that won't be available to a planner), with each class contributing `QueueRev[i]`;

- `QueueCost[i]` is the projected cost to be incurred by the patients left at the end of the year from class i

Finally, `fracallocate` is calculated as the estimated fraction of revenue for patients that end one year that will be of use the following year (using this year's value of `CapMiss`) so that

```
If CapMiss ≤ 0 then fracallocate=1; else
    if CapMiss-EstRev ≥ 0 then fracallocate=0; else
        fracallocate = CapMiss/EstRev.
```

The estimated terminal value for class i is then

```
(fracallocate*QueueRev[i]-QueueCost[i])/NumPatients[i]
```

where `NumPatients[i]` is the number of class i left at the end of the year and `fracallocate` becomes our estimate for ψ. Using the base case parameter values, this yields an estimated ψ of 0.15.

Because the number of customers left at the end of the year, and hence the calculated terminal values, is likely to be quite variable we update the terminal values using an exponential smoothing type algorithm. In particular, if `ALPHA` is the exponential smoothing parameter (set equal to 0.1 in our tests) then we calculate the new terminal value as

```
ALPHA*(fracallocate*QueueRev[i]-QueueCost[i])/NumPatients[i]+(1-ALPHA)*Terminal[i]
```

where `Terminal[i]` is the previous terminal value for class i patients.

D References

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