Appendices to

"Social Responsibility Auditing in Supply Chain Networks"

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Appendix A: Symbols Used

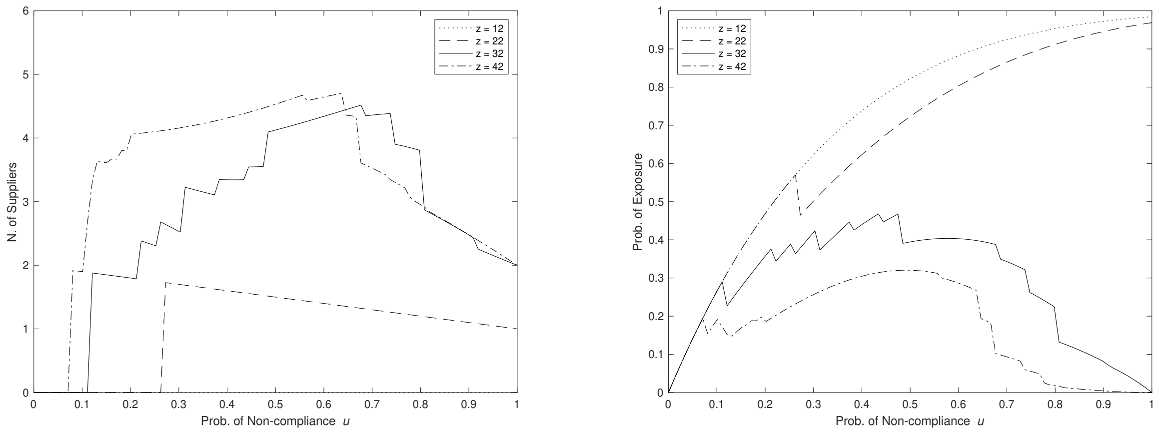
$\nabla(\gamma,i)$	$=\pi(\gamma)-\pi(\gamma\ominus i)$ for $i\in U_\gamma$ where $\gamma\in\Gamma$, the production value of supplier i in state γ
$\oplus,\ominus$	operators that map a state-unaudited-supplier pair to a state; used in defining dynamic program
a	cost of auditing a supplier
A, B	tier-1 firms
α,β	parameters of the demand function from customers downstream to the buyer
C	the buyer
C_i	strategy space of firm i
$D_g(i)$	set of dependents of supplier i in supply network g
e_{A}, e_{B}	an exclusive supplier to firm A and that to firm B, used in notation for AD and AR decisions
g	supply network
$g_\emptyset$	the null supply network
$\gamma=(g,U)$	state in the auditing phase
Γ	state space of the auditing phase
Γ_T	set of terminal states
$p_{(0)}$	selling price of buyer
$p_{(1)}$	selling price of tier-1 firms to the buyer
$p_{(2)i}$	selling price of tier-2 suppliers to tier-1 firm i
$\pi(\gamma)$	buyer's production profit in state γ
π_i	profit of firm i from production activity
q_i	total quantity produced by firm i
r	cost of rectifying a noncompliant supplier
$R^+(\gamma)$	set of states reachable from state γ
$s_{j,i}$	quantity supplier j produces for downstream firm i
$	a shared supplier, used in notation for AD and AR decisions
S_{A}, S_{B}	set of exclusive suppliers to tier-1 firm A and that to firm B
S_{AB}	set of shared suppliers
S_g	set of suppliers in supply network g
$S(k)$	set of suppliers in tier $k=1,2$
u	probability that an unaudited supplier is noncompliant
U_γ	set of unaudited suppliers in state γ
$\overline{U}$	union of sets of unaudited suppliers in any state in Γ
V	value function in auditing phase
V^*	optimal value function in auditing phase
$\widetilde{V}$	state value function in auditing phase
$\widetilde{V}^*$	optimal state value function in auditing phase
v_k	unit production cost in tier k
v_T	sum of production costs per unit across tiers
w	probability that violation at a supplier will be exposed, given that it is noncompliant
X_γ	set of admissible actions at state γ
$\overline{X}$	union of sets of admissible actions in any state in Γ
ξ	auditing policy
Ξ	set of all auditing policies
z	cost to the buyer of an exposed violation
Z	set of state-unaudited-supplier pairs
$\zeta(\gamma)$	expected total penalty from violations on state γ

Appendix B: Effect of Probability of Noncompliance

What is the role of the ex ante probability of noncompliance u in the auditing activity and the level of risk in the supply network resulting from the auditing phase? We consider a state with supply network $g=(\{\text{A},\text{B}\},\{1,2\},\{4\},\{3\})$ (as in Figure 1) in which all suppliers are unaudited, with values of parameters

Figure B.1 Auditing and Risk as Probability of Noncompliance Varies

(a) Expected Number of Suppliers Audited (b) Probability of Exposure in Production Phase



State $\gamma = (g, U)$ where $g = (\{\text{A}, \text{B}\}, \{1, 2\}, \{4\}, \{3\})$ (Figure 1) and $U = S_g$. Parameters $\alpha = 100$, $\beta = 10$, $v_T = 2$, $a = 1$, $r = 20$, $w = 0.5$. (In the left panel, the graph for $z = 12$ coincides with the horizontal axis.)

$\alpha = 100$, $\beta = 10$, $v_T = 2$, $a = 1$, $r = 20$, and $w = 0.5$. Figure B.1a shows the *expected* number of suppliers to be audited throughout the auditing phase as u increases from 0 to 1 for various values of penalty z; this expectation is taken over all possible sample paths of whether any supplier passes or fails an audit under the optimal auditing policy. Figure B.1b shows the corresponding *expected* probability of the exposure of violation at any remaining unaudited suppliers in the supply network after the auditing phase. Given state γ at the conclusion of the auditing phase, we calculate the probability by $1 - (1 - uw)^{|U_\gamma|}$ where uw is the probability of an unaudited supplier being noncompliant and subsequently exposed and U_γ is the set of any unaudited suppliers at state γ (these are the only suppliers which could possibly violate in the production phase). The set U_γ is the culmination of the path-dependent auditing process.

In Figures B.1a and B.1b, a jump from one smooth segment on a curve to the next smooth segment represents a shift in the auditing policy. For low values of z (e.g., $z \leqslant 12$) the buyer conducts no audits at all regardless of the probability of noncompliance; the number of suppliers to be audited remains zero and the network's probability of exposure increases monotonically in u. With higher z the buyer starts to audit once u reaches a threshold, which decreases as z gets larger. For example, compare the $z = 22$ to the $z = 12$ curves: for $z = 22$ when $u \geqslant 0.27$ the buyer is sufficiently concerned about noncompliance that some auditing will occur, resulting in a corresponding drop in the probability of exposure, relative to the $z = 12$ curve. Overall as u increases the expected number of suppliers audited first shows an upward trend, reflecting the buyer's greater concern of the potential penalty from violation, leading to more audits. The expected number of audited firms trends downward, however, as u increases further. With a higher probability of noncompliance the business becomes too risky so the buyer turns to auditing the tier-1 firms directly. The buyer expects to drop these tier-1 firms, along with their tier-2 dependents if the tier-1 firms turn out to be noncompliant, thus avoiding the cost of conducting those tier-2 audits (highly likely to be noncompliant). As u approaches 1, the expected number of audits approaches 2, because the buyer audits firms A and B (which are very likely to fail the audits), thus dropping them and killing the business. Figure B.1b reflects the same effects: the probability of exposure in the network first exhibits an overall upward trend with increasing u but eventually falls to zero as the buyer drops both firms A and B, and consequently the entire supply network, thus avoiding risk entirely.

Neither figure shows a generally monotonic pattern. As u increases, driven by the higher probability of noncompliance the buyer adopts an increasingly aggressive approach to auditing, concomitantly reducing in the probability of exposure (Figure B.1b). Ultimately, such auditing may be exhaustive (leaving no firm unaudited) to ensure full compliance or to extinguish the business.

Appendix C: Proofs for the Production Phase

C.1. Existence and Uniqueness of Equilibrium

PROPOSITION C.1. *Given the buyer's input price $p_{(1)}$, there exists a unique optimal quantity q_{C}^* which solves the buyer's problem P_0. Moreover, the resulting inverse demand function faced by the tier-1 firms is*

$$p^*_{(1)}(q_{\text{C}}) = \alpha - v_0 - 2\beta q_{\text{C}}. \tag{C.1}$$

Proof. Substitute (6) into (7) and differentiate to get

$$\frac{\partial \pi_{\text{C}}}{\partial q_{\text{C}}} = -\beta q_{\text{C}} + (\alpha - \beta q_{\text{C}} - v_0 - p_{(1)}) \tag{C.2}$$

$$\frac{\partial^2 \pi_{\text{C}}}{\partial q_{\text{C}}^2} = -2\beta. \tag{C.3}$$

Since $\beta > 0$, (C.3) implies that π_{C} is strictly concave. Hence a quantity q_{C} maximizes π_{C} if and only if it sets $\frac{\partial \pi_{\text{C}}}{\partial q_{\text{C}}} = 0$ in (C.2); the unique such q_{C} is given by

$$q_{\text{C}}^* = \frac{\alpha - v_0}{2\beta} - \frac{p_{(1)}}{2\beta}. \tag{C.4}$$

We rewrite it in the form of an inverse demand function to obtain (C.1). □

PROPOSITION C.2. *Given the tier-1 vector of input prices $\mathbf{p}_{(2)} = (p_{(2)i})_{i \in S(1)}$, there exists a unique equilibrium in pure strategies $\mathbf{q}^*_{(1)}$ of the game P_1. Moreover, the resulting inverse demand function faced by the tier-2 firms supplying firm i is (for $i \in S(1)$)*

$$p^*_{(2)i}(\mathbf{q}_{(1)}) = \alpha - v_0 - v_1 - 4\beta q_i - 2\beta \sum_{i' \in S(1)\setminus\{i\}} q_{i'}. \tag{C.5}$$

Proof. Substitute (C.1) into (8) to get tier-1 supplier i's profit

$$\pi_i = (\alpha - v_0 - 2\beta q_{\text{C}} - v_1 - p_{(2)i}) q_i. \tag{C.6}$$

Substitute (9) into (C.6) to get

$$\pi_i = \left(\alpha - v_0 - v_1 - p_{(2)i} - 2\beta \sum_{j \in S(1)} q_j \right) q_i. \tag{C.7}$$

Then

$$\frac{\partial \pi_i}{\partial q_i} = \alpha - v_0 - v_1 - p_{(2)i} - 4\beta q_i - 2\beta \sum_{j \in S(1)\setminus\{i\}} q_j \tag{C.8}$$

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -4\beta. \tag{C.9}$$

Given $\beta > 0$, (C.9) implies π_i is strictly concave in q_i. Given any other tier-1 firm's decision, a quantity q_i maximizes π_i if and only if it sets $\frac{\partial \pi_i}{\partial q_i} = 0$ in (C.8); the unique such q_i is

$$q_i = -\frac{1}{2}\left(\sum_{j \in S(1)\setminus\{i\}} q_j\right) + \frac{\alpha - v_0 - v_1 - p_{(2)i}}{4\beta}. \tag{C.10}$$

Hence a strategy profile $\mathbf{q}_{(1)} = (q_i)_{i \in S(1)}$ is an equilibrium of the tier-1 firms' game P_1 if and only if it solves the system of linear equations (C.10) for all $i \in S(1)$.

If $|S(1)| = 1$, let $i \in S(1)$, then it is clear that $\mathbf{q}_{(1)} = q_i = \frac{\alpha - v_0 - v_1 - p_{(2)i}}{4\beta}$ is the unique (degenerate) equilibrium of the game P_1. If $|S(1)| = 2$, i.e., $S(1) = \{\text{A}, \text{B}\}$, we write the system (C.10) as

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \mathbf{q}_{(1)} = \frac{\alpha - v_0 - v_1}{4\beta} - \frac{1}{4\beta}\begin{bmatrix} p_{(2)\text{A}} \\ p_{(2)\text{B}} \end{bmatrix}. \tag{C.11}$$

Clearly the matrix $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ is invertible; hence the system (C.10) has a unique solution, which is the unique equilibrium of the game P_1. We rewrite (C.10) in the form of an inverse demand function to obtain (C.5). □

Proposition C.3. *(a) A tier-2 supplier j's profit π_j is strictly concave in $\mathbf{s}_j$.*

(b) There exists a unique equilibrium of the game P_2 of Cournot competition among tier-2 suppliers in the first stage of the production phase.

(c) Given the inverse demand function (C.5) *from tier-1 firm i that its tier-2 suppliers collectively receive, for $j \in S_i$, $i \in S(1)$,*

$$\frac{\partial \pi_j(\mathbf{s}_{(2)})}{\partial s_{j,i}} = -4\beta s_{j,i} + \left[\alpha - v_T - 4\beta \sum_{j' \in S_i \cup S_{\text{AB}}} s_{j',i} - 2\beta \sum_{i' \in S(1)\setminus\{i\}} \left(\sum_{j' \in S_{i'} \cup S_{\text{AB}}} s_{j',i'}\right)\right] \tag{C.12}$$

and for $j \in S_{\text{AB}}$ and $i \in S(1)$,

$$\frac{\partial \pi_j(\mathbf{s}_{(2)})}{\partial s_{j,i}} = -4\beta s_{j,i} + \left[\alpha - v_T - 4\beta \sum_{j' \in S_i \cup S_{\text{AB}}} s_{j',i} - 2\beta \sum_{i' \in S(1)\setminus\{i\}} \left(\sum_{j' \in S_{i'} \cup S_{\text{AB}}} s_{j',i'}\right)\right] - 2\beta \sum_{i' \in S(1)\setminus\{i\}} s_{j,i'}. \tag{C.13}$$

Proof. (Part (c)). For exclusive supplier $j \in S_i$, $i \in S(1)$, substitute (C.5) into (11), replace $v_0 + v_1 + v_2$ with v_T, and then substitute (13) to get

$$\pi_j = \left(\alpha - v_T - 4\beta q_i - 2\beta \sum_{i' \in S(1)\setminus\{i\}} q_{i'}\right) s_{j,i} \tag{C.14}$$

$$= \left[\alpha - v_T - 4\beta \sum_{j' \in S_i \cup S_{\text{AB}}} s_{j',i} - 2\beta \sum_{i' \in S(1)\setminus\{i\}} \left(\sum_{j' \in S_{i'} \cup S_{\text{AB}}} s_{j',i'}\right)\right] s_{j,i}. \tag{C.15}$$

Differentiate (C.15) with respect to $s_{j,i}$ to get (C.12). For shared supplier $j \in S_{\text{AB}}$, substitute (C.5) into (12), replace $v_0 + v_1 + v_2$ with v_T, and then substitute (13) to get

$$\pi_j = \sum_{i \in S(1)} \left(\alpha - v_T - 4\beta q_i - 2\beta \sum_{i' \in S(1)\setminus\{i\}} q_{i'}\right) s_{j,i} \tag{C.16}$$

$$= \sum_{i \in S(1)} \left[\alpha - v_T - 4\beta \sum_{j' \in S_i \cup S_{\text{AB}}} s_{j',i} - 2\beta \sum_{i' \in S(1)\setminus\{i\}} \left(\sum_{j' \in S_{i'} \cup S_{\text{AB}}} s_{j',i'}\right)\right] s_{j,i}. \tag{C.17}$$

Differentiate (C.17) with respect to $s_{j,i}$, $i \in S(1)$, to get (C.13).

(Part (a)). For exclusive supplier $j \in S_i$, $i \in S(1)$, differentiate (C.12) with respect to $s_{j,i}$ to get

$$\frac{\partial^2 \pi_j}{\partial s_{j,i}^2} = -8\beta. \tag{C.18}$$

Therefore π_j is strictly concave in $\mathbf{s}_j = s_{j,i}$. For shared supplier $j \in S_{\text{AB}}$, differentiate (C.13) with respect to $s_{j,i}$ and with respect to $s_{j,i'}$ for $i' \in S(1)\backslash\{i\}$ to get

$$\frac{\partial^2 \pi_j}{\partial s_{j,i}^2} = -8\beta \quad \text{and} \quad \frac{\partial^2 \pi_j}{\partial s_{j,i'} \partial s_{j,i}} = -4\beta. \tag{C.19}$$

Hence the Hessian of π_j with respect to $\mathbf{s}_j$ is $\begin{bmatrix} -8\beta & -4\beta \\ -4\beta & -8\beta \end{bmatrix}$ which, given that $\beta > 0$, can be easily verified to be negative definite. Therefore π_j is strictly concave in $\mathbf{s}_j = \{s_{j,i}\}_{i \in S(1)}$.

(Part (b)). (Existence.) The strategy space C_j of tier-2 supplier j is a nonempty compact convex subset of $\mathbb{R}$ (if j is an exclusive supplier) or $\mathbb{R}^2$ (if j is a shared supplier). The payoff function π_j of supplier j (C.15) (for exclusive supplier j) or (C.17) (for shared supplier j) is continuous in the strategy profile $(\mathbf{s}_j)_{j \in S(2)}$ and strictly concave, hence quasi-concave, in supplier j's own strategy $\mathbf{s}_j$. By Proposition 20.3 in Osborne and Rubinstein (1994), there exists a pure-strategy equilibrium of the game among tier-2 suppliers in the first stage.

(Uniqueness.) We use the method due to Rosen (1965). As we have seen, the strategy space C_j of supplier j is convex, closed, and bounded. π_j is continuous in the strategy profile and concave in supplier j's strategy. Label a tier-1 firm A and, if there is a second one, label it B. Label $S(2) = \{1, \ldots, n\}$, where $n = |S(2)|$, such that $\{1, \ldots, t_{\text{A}}\}$ are tier-1 firm A's exclusive suppliers, $\{t_{\text{A}} + 1, \ldots, t_{\text{A}} + t_{\text{AB}}\}$ are the shared suppliers, and $\{t_{\text{A}} + t_{\text{AB}} + 1, \ldots, n\}$ are tier-1 firm B's exclusive suppliers (any of the subsets could be empty, but at least $S(2)$ is nonempty, i.e., $n > 0$). Let $\mathbf{x} = (\mathbf{s}_j)_{j \in S(2)}$. We choose $\mathbf{r} = \boldsymbol{\iota}_n = (1, \ldots, 1)^\intercal \in \mathbb{R}^n$ as the weights for the payoff functions in $\sigma(\mathbf{x}, \mathbf{r})$ in Rosen (1965). Then the pseudogradient of $\sigma(\mathbf{x}, \mathbf{r}) = \sigma(\mathbf{x}, \boldsymbol{\iota}_n)$ is

$$g(\mathbf{x}, \boldsymbol{\iota}_n) = \left(\frac{\partial \pi_1}{\partial s_{1,\text{A}}}, \ldots, \frac{\partial \pi_{t_{\text{A}}}}{\partial s_{t_{\text{A}},\text{A}}}, \frac{\partial \pi_{t_{\text{A}}+1}}{\partial s_{t_{\text{A}}+1,\text{A}}}, \frac{\partial \pi_{t_{\text{A}}+1}}{\partial s_{t_{\text{A}}+1,\text{B}}}, \ldots, \frac{\partial \pi_{t_{\text{A}}+t_{\text{AB}}}}{\partial s_{t_{\text{A}}+t_{\text{AB}},\text{A}}}, \frac{\partial \pi_{t_{\text{A}}+t_{\text{AB}}}}{\partial s_{t_{\text{A}}+t_{\text{AB}},\text{B}}}, \frac{\partial \pi_{t_{\text{A}}+t_{\text{AB}}+1}}{\partial s_{t_{\text{A}}+t_{\text{AB}}+1,\text{B}}}, \ldots, \frac{\partial \pi_n}{\partial s_{n,\text{B}}} \right)^\intercal \in \mathbb{R}^{t_{\text{A}}+2t_{\text{AB}}+t_{\text{B}}} \tag{C.20}$$

By (C.18), (C.19), (C.15), and (C.17), the Jacobian $G(\mathbf{x}, \boldsymbol{\iota}_n)$ of $g(\mathbf{x}, \boldsymbol{\iota}_n)$ with respect to $\mathbf{x}$ is equal to $-2\beta\Lambda(t_{\text{A}}, t_{\text{B}}, t_{\text{AB}})$, a symmetric matrix. Hence $G(\mathbf{x}, \boldsymbol{\iota}_n) + (G(\mathbf{x}, \boldsymbol{\iota}_n))^\intercal = -4\beta\Lambda(t_{\text{A}}, t_{\text{B}}, t_{\text{AB}})$, which is negative definite for any $\mathbf{x} \in \prod_{j \in S(2)} C_j$ by Lemma I.3 in Appendix I and that $\beta > 0$. By Theorem 6 in Rosen (1965), $\sigma(\mathbf{x}, \boldsymbol{\iota}_n)$ is diagonally strictly concave. By Theorem 2 in Rosen (1965), the equilibrium of the game among tier-2 suppliers in the first stage of the production phase is unique. □

Proof of Theorem 1. By Proposition C.3, there exists a unique equilibrium $(\mathbf{s}_j^*)_{j \in S(2)}$ of the game in the first stage among tier-2 supplier; let $(p_{(2)i}^*)_{i \in S(1)}$ be the resulting selling prices of the tier-2 suppliers. Given $(p_{(2)i}^*)_{i \in S(1)}$, by Proposition C.2, there exists a unique equilibrium $\mathbf{q}_{(1)}^*$ of the game in the second stage among tier-1 firms; let $p_{(1)}^*$ be the resulting selling price of the tier-1 firms. Given $p_{(1)}^*$, by Proposition C.1, there exists a unique optimal solution q_{C}^* to the buyer's problem in the first stage; let $p_{(0)}^*$ be the resulting selling price of the buyer. Hence the tuple of prices and quantities $\left(p_{(0)}^*, p_{(1)}^*, (p_{(2)i}^*)_{i \in S(1)}, q_{\text{C}}^*, \mathbf{q}_{(1)}^*, (\mathbf{s}_j^*)_{j \in S(2)}\right)$ is the unique production phase equilibrium. □

C.2. Equilibrium Characterization

We begin with a simple relationship between the buyer's equilibrium production quantity and profit:

PROPOSITION C.4. *The buyer's equilibrium profit is* $\pi_{\mathrm{C}}^* = \beta \cdot (q_{\mathrm{C}}^*)^2$.

Proof. Substitute (6) for $p_{(0)}$ and (C.1) for $p_{(1)}$ in (7). $\square$

In equilibrium the buyer's production profit depends only on the buyer's quantity of production q_{C}^*, which is also the total quantity produced by the supply network.

The following proposition provides closed-form expressions for the equilibrium quantities. We define functions $L, s_e, s_s, \bar{q} : \mathbb{R}^3 \to \mathbb{R}$ to facilitate representation of the equilibrium quantities.

$$L(x_1,x_2,x_3) = 4x_1 + 4x_2 + 8x_3 + 3x_1x_2 + 4x_1x_3 + 4x_2x_3 + 4x_3^2 + 4 \tag{C.21}$$

$$s_e(x_1,x_2,x_3) = \frac{1}{2}\frac{\alpha - v_T}{\beta}\left(\frac{x_2 + 2x_3 + 2}{L(x_1,x_2,x_3)}\right) \tag{C.22}$$

$$s_s(x_1,x_2,x_3) = \frac{1}{3}\frac{\alpha - v_T}{\beta}\left(\frac{-x_1 + 2x_2 + 2x_3 + 2}{L(x_1,x_2,x_3)}\right) \tag{C.23}$$

$$\bar{q}_{(1)}(x_1,x_2,x_3) = \frac{1}{6}\frac{\alpha - v_T}{\beta}\left(\frac{4x_3^2 + 4x_1x_3 + 4x_2x_3 + 4x_3 + 6x_1 + 3x_1x_2}{L(x_1,x_2,x_3)}\right) \tag{C.24}$$

$$\bar{q}(x_1,x_2,x_3) = \frac{1}{3}\frac{\alpha - v_T}{\beta}\left(\frac{3x_1 + 3x_2 + 4x_3 + 4x_1x_3 + 4x_2x_3 + 3x_1x_2 + 4x_3^2}{L(x_1,x_2,x_3)}\right). \tag{C.25}$$

PROPOSITION C.5. (*a*) *If* $t_{\mathrm{A}} \leqslant 2t_{\mathrm{B}} + 2t_{\mathrm{AB}} + 2$, *in equilibrium:*

i. *Exclusive supplier* $j \in S_i$ *to tier-1 firm* $i \in S(1)$ *chooses supply quantity* $s_{j,i}^* = s_e(t_i, t_{-i}, t_{\mathrm{AB}})$ *where* $-i \in \{\mathrm{A},\mathrm{B}\}\backslash\{i\}$;

ii. *Shared supplier* $j \in S_{\mathrm{AB}}$ *chooses supply quantities* $s_{j,\mathrm{A}}^* = s_s(t_{\mathrm{A}}, t_{\mathrm{B}}, t_{\mathrm{AB}})$ *and* $s_{j,\mathrm{B}}^* = s_s(t_{\mathrm{B}}, t_{\mathrm{A}}, t_{\mathrm{AB}})$;

iii. *Tier-1 firm* $i \in S(1)$ *chooses supply quantity* $q_i^* = t_i s_e(t_i, t_{-i}, t_{\mathrm{AB}}) + t_{\mathrm{AB}} s_s(t_i, t_{-i}, t_{\mathrm{AB}}) = \bar{q}_{(1)}(t_i, t_{-i}, t_{\mathrm{AB}})$ *where* $-i \in \{\mathrm{A},\mathrm{B}\}\backslash\{i\}$;

iv. *The total quantity the supply network produces is*

$$q_{\mathrm{C}}^* = t_{\mathrm{A}} s_e(t_{\mathrm{A}}, t_{\mathrm{B}}, t_{\mathrm{AB}}) + t_{\mathrm{B}} s_e(t_{\mathrm{B}}, t_{\mathrm{A}}, t_{\mathrm{AB}}) + t_{\mathrm{AB}}(s_s(t_{\mathrm{A}}, t_{\mathrm{B}}, t_{\mathrm{AB}}) + s_s(t_{\mathrm{B}}, t_{\mathrm{A}}, t_{\mathrm{AB}})) = \bar{q}(t_{\mathrm{A}}, t_{\mathrm{B}}, t_{\mathrm{AB}}). \tag{C.26}$$

(*b*) *If* $t_{\mathrm{A}} \geqslant 2t_{\mathrm{B}} + 2t_{\mathrm{AB}} + 2$, *in equilibrium:*

i. *Firm* A*'s exclusive supplier* $j \in S_{\mathrm{A}}$ *chooses supply quantity* $s_{j,\mathrm{A}}^* = s_e(t_{\mathrm{A}}, t_{\mathrm{B}} + t_{\mathrm{AB}}, 0)$;

ii. *Firm* B*'s exclusive supplier* $j \in S_{\mathrm{B}}$ *chooses supply quantity* $s_{j,\mathrm{B}}^* = s_e(t_{\mathrm{B}} + t_{\mathrm{AB}}, t_{\mathrm{A}}, 0)$;

iii. *Shared supplier* $j \in S_{\mathrm{AB}}$ *chooses supply quantities* $s_{j,\mathrm{A}}^* = 0$ *and* $s_{j,\mathrm{B}}^* = s_e(t_{\mathrm{B}} + t_{\mathrm{AB}}, t_{\mathrm{A}}, 0)$;

iv. *Firm* A *chooses supply quantity* $q_{\mathrm{A}}^* = t_{\mathrm{A}} s_e(t_{\mathrm{A}}, t_{\mathrm{B}} + t_{\mathrm{AB}}, 0) = \bar{q}_{(1)}(t_{\mathrm{A}}, t_{\mathrm{B}} + t_{\mathrm{AB}}, 0)$;

v. *Firm* B *chooses supply quantity* $q_{\mathrm{B}}^* = (t_{\mathrm{B}} + t_{\mathrm{AB}}) s_e(t_{\mathrm{B}} + t_{\mathrm{AB}}, t_{\mathrm{A}}, 0) = \bar{q}_{(1)}(t_{\mathrm{B}} + t_{\mathrm{AB}}, t_{\mathrm{A}}, 0)$;

vi. *The total quantity the supply network produces is*

$$q_{\mathrm{C}}^* = t_{\mathrm{A}} s_e(t_{\mathrm{A}}, t_{\mathrm{B}} + t_{\mathrm{AB}}, 0) + (t_{\mathrm{B}} + t_{\mathrm{AB}}) s_e(t_{\mathrm{B}} + t_{\mathrm{AB}}, t_{\mathrm{A}}, 0) = \bar{q}(t_{\mathrm{A}}, t_{\mathrm{B}} + t_{\mathrm{AB}}, 0). \tag{C.27}$$

Proof. Tier-2 supplier j's problem is

$$(P_{(2)j}) \qquad \max \quad \pi_j(\mathbf{s}_{(2)}) \tag{C.28}$$

$$\text{subject to} \quad \mathbf{s}_j \geqslant 0. \tag{C.29}$$

By Proposition C.3(a), $\pi_j(\mathbf{s}_{(2)})$ is concave in $\mathbf{s}_j$. With merely the nonnegativity constraints, constraint qualification always holds. Therefore the Karush-Kuhn-Tucker (KKT) conditions

$$\frac{\partial \pi_j(\mathbf{s}^*_{(2)})}{\partial s_{j,i}} \leqslant 0, \quad \text{with equality if } s^*_{j,i} > 0, \qquad (\text{if } j \in S_i, i \in S(1); \text{ or for } i \in S(1) \text{ if } j \in S_{\text{AB}}) \tag{C.30}$$

are necessary and sufficient for $\mathbf{s}^*_j \geqslant 0$ to be a global maximizer. Given $(\mathbf{s}^*_{j'})_{j' \in S(2)\setminus\{j\}}$, if $\mathbf{s}^*_j$ solves $P_{(2)j}$, then $\mathbf{s}^*_j$ is a best response to $(\mathbf{s}^*_{j'})_{j' \in S(2)\setminus\{j\}}$. If for every $j \in S(2)$, $\mathbf{s}^*_j$ is a best response to $(\mathbf{s}^*_{j'})_{j' \in S(2)\setminus\{j\}}$, then $\mathbf{s}^*_{(2)}$ is an equilibrium in pure strategies of P_2.

(Case (a)). We note $s_e(x_1, x_2, x_3) > 0, \forall x_1, x_2, x_3 \geqslant 0$; hence the supply quantity of every exclusive supplier, as defined using s_e, is positive. If there exists a shared supplier j, then by the assumption $t_{\text{A}} \leqslant 2t_{\text{B}} + 2t_{\text{AB}} + 2$ and the expression in (C.23), $s^*_{j,\text{A}} = s_s(t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}) \geqslant 0$. Since $t_{\text{A}} \geqslant t_{\text{B}}$, $s^*_{j,\text{B}} = s_s(t_{\text{B}}, t_{\text{A}}, t_{\text{AB}}) > 0$. $\mathbf{s}^*_j$ as defined is nonnegative for every $j \in S(2)$.

Substituting supply quantities in parts (a)i–(a)ii into (C.12) and (C.13), following some algebra, we verify that $\frac{\partial \pi_j(\mathbf{s}^*_{(2)})}{\partial s^*_{j,i}} = 0$ for $i \in S(1)$ and $j \in S_i \cup S_{\text{AB}}$. Therefore $\mathbf{s}^*_{(2)}$ satisfies the nonnegativity constraints and the KKT conditions (C.30) for every tier-2 supplier $j \in S(2)$. Hence $\mathbf{s}^*_{(2)}$ is an equilibrium of P_2.

We verify part (a)iii by substituting the values of $s^*_{j,i}$ in parts (a)i and (a)ii into (13) for the corresponding quantities, and substituting (C.22), (C.23), and (C.24). We verify (C.26) by substituting (13) into (9), then substituting the values of $s^*_{j,i}$ in parts (a)i and (a)ii for the corresponding quantities.

(Case (b)). Except for $s^*_{j,\text{A}} = 0$ for a shared supplier j, the supply quantities in parts (b)i–(b)iii are defined using s_e and, as such, positive. Therefore $\mathbf{s}^*_j$ as defined is nonnegative for every $j \in S(2)$.

Analogous to case (a), substituting supply quantities in parts (b)i–(b)iii into (C.12) and (C.13), following some algebra, we verify that $\frac{\partial \pi_j(\mathbf{s}^*_{(2)})}{\partial s^*_{j,i}} = 0$ for exclusive supplier $j \in S_i$, $i \in S(1)$, and for shared supplier $j \in S_{\text{AB}}$ and $i = \text{B}$. We only need to verify that $\frac{\partial \pi_j(\mathbf{s}^*_{(2)})}{\partial s_{j,\text{A}}} \leqslant 0$ for every shared supplier $j \in S_{\text{AB}}$ to verify the KKT conditions (C.30). We substitute the supply quantities $\mathbf{s}^*_{(2)}$ into (C.13) to find

$$\frac{\partial \pi_j(\mathbf{s}^*_{(2)})}{\partial s_{j,\text{A}}} = a - v_T - 4\beta t_{\text{A}} s_e(t_{\text{A}}, t_{\text{B}} + t_{\text{AB}}, 0) - 2\beta(t_{\text{B}} + t_{\text{AB}}) s_e(t_{\text{B}} + t_{\text{AB}}, t_{\text{A}}, 0) - 2\beta s_e(t_{\text{B}} + t_{\text{AB}}, t_{\text{A}}, 0) \tag{C.31}$$

$$= a - v_T - 4\beta t_{\text{A}} s_e(t_{\text{A}}, t_{\text{B}} + t_{\text{AB}}, 0) - 2\beta(t_{\text{B}} + t_{\text{AB}} + 1) s_e(t_{\text{B}} + t_{\text{AB}}, t_{\text{A}}, 0). \tag{C.32}$$

$L(x_1, x_2, x_3)$ is symmetric in x_1 and x_2 in the sense that $L(x_1, x_2, x_3) = L(x_2, x_1, x_3)$. Let $\widehat{L} = L(t_{\text{A}}, t_{\text{B}} + t_{\text{AB}}, 0) = L(t_{\text{B}} + t_{\text{AB}}, t_{\text{A}}, 0)$. Substitute the definition of s_e into (C.32) to get

$$\frac{\partial \pi_j(\mathbf{s}^*_{(2)})}{\partial s_{j,\text{A}}} = a - v_T - 2(\alpha - v_T)\frac{t_{\text{A}}(t_{\text{B}} + t_{\text{AB}} + 2)}{\widehat{L}} - (\alpha - v_T)\frac{(t_{\text{A}} + 2)(t_{\text{B}} + t_{\text{AB}} + 1)}{\widehat{L}} \tag{C.33}$$

$$= (a - v_T)\left[1 - \frac{1}{\widehat{L}}(3t_{\text{A}}t_{\text{B}} + 3t_{\text{A}}t_{\text{AB}} + 5t_{\text{A}} + 2t_{\text{B}} + 2t_{\text{AB}} + 2)\right] \tag{C.34}$$

$$= (a - v_T)\frac{1}{\widehat{L}}(-t_{\text{A}} + 2t_{\text{B}} + 2t_{\text{AB}} + 2) \tag{C.35}$$

which is nonpositive by the assumption $\alpha \geqslant v_T$ and the premise $t_{\text{A}} \geqslant 2t_{\text{B}} + 2t_{\text{AB}} + 2$.

We verify parts (b)iv and (b)v by substituting the values of $s^*_{j,i}$ in parts (b)i–(b)iii into (13) for the corresponding quantities. We verify (C.27) by substituting (13) into (9) then substituting the values of $s^*_{j,i}$ in parts (b)i–(b)iii for the corresponding quantities. □

C.3. Comparative Statics

In this section, we consider only the case $\alpha > v_T$. (If $\alpha = v_T$, by Proposition C.5, every supply quantity is zero.) All comparative statics results, except those on q^*_{C} and π^*_{C}, are applicable only to non-null supply networks. To facilitate the proofs we introduce an alternative notation of a supply network in terms of t_{A}, the number of exclusive suppliers to firm A, t_{B} the number of exclusive suppliers to firm B, and t_{AB} the number of shared suppliers. Specifically, we define an operator $\langle\cdot\rangle : \mathbb{N}_0^3 \to G$ by

$$\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle = \begin{cases} (\{\text{A},\text{B}\}, \{1,\ldots,t_{\text{A}}\}, \{t_{\text{A}}+1,\ldots,t_{\text{A}}+t_{\text{B}}\}, \\ \qquad \{t_{\text{A}}+t_{\text{B}}+1,\ldots,t_{\text{A}}+t_{\text{B}}+t_{\text{AB}}\}), & \text{if } t_{\text{A}}+t_{\text{AB}}>0 \text{ and } t_{\text{B}}+t_{\text{AB}}>0 \\ (\{\text{A}\}, \{1,\ldots,t_{\text{A}}+t_{\text{AB}}\}, \emptyset, \emptyset), & \text{if } t_{\text{A}}+t_{\text{AB}}>0 \text{ and } t_{\text{B}}+t_{\text{AB}}=0\ . \\ (\{\text{B}\}, \emptyset, \{1,\ldots,t_{\text{B}}+t_{\text{AB}}\}, \emptyset), & \text{if } t_{\text{A}}+t_{\text{AB}}=0 \text{ and } t_{\text{B}}+t_{\text{AB}}>0 \\ g_\emptyset, & \text{if } t_{\text{A}}+t_{\text{AB}}=0 \text{ and } t_{\text{B}}+t_{\text{AB}}=0 \end{cases} \tag{C.36}$$

Let f be a variable that arises from the production phase equilibrium (quantity, margin, profit, price, market share). We denote by $f(g) = f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle$ the value of this variable in supply network $g = \langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle$. We further denote $\Delta_1 f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle = f\langle t_{\text{A}}+1, t_{\text{B}}, t_{\text{AB}}\rangle - f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle$, $\Delta_2 f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle = f\langle t_{\text{A}}, t_{\text{B}}+1, t_{\text{AB}}\rangle - f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle$, and $\Delta_3 f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle = f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}+1\rangle - f\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle$.

Proof of Theorem 2. We begin by noting a few relationships between the equilibrium variables. By the definition of $p_{(0)}$ and (C.1),

$$m^*_{\text{C}} = (\alpha - \beta q^*_{\text{C}}) - (\alpha - v_0 - 2\beta q^*_{\text{C}}) - v_0 = \beta q^*_{\text{C}}. \tag{C.37}$$

By Proposition C.4, $\pi^*_{\text{C}} = \beta (q^*_{\text{C}})^2$. Therefore, q^*_{C}, m^*_{C}, and π^*_{C} always change in the same direction, which is opposite to the change in $p^*_{(1)}$ by (C.1). Hence the direction of the change in any one of the four variables determines those of the other three. Let $i \in S(1)$. By (C.5),

$$p^*_{(2)i} = \alpha - v_0 - v_1 - 4\beta q^*_i - 2\beta(q^*_{\text{C}} - q^*_i) = \alpha - v_0 - v_1 - 2\beta q^*_i - 2\beta q^*_{\text{C}}. \tag{C.38}$$

Then by (C.1),

$$m^*_i = (\alpha - v_0 - 2\beta q^*_{\text{C}}) - v_1 - (\alpha - v_0 - v_1 - 2\beta q^*_i - 2\beta q^*_{\text{C}}) = 2\beta q^*_i. \tag{C.39}$$

By (8),

$$\pi^*_i = m^*_i q^*_i = 2\beta (q^*_i)^2. \tag{C.40}$$

Therefore, q^*_i, m^*_i, and π^*_i always change in the same direction. Finally, $\rho^*_{\text{A}} = 1 - \rho^*_{\text{B}}$.

Given the characterization of the equilibrium in Proposition C.5, we directly calculate the change $\Delta_k f(g)$ in each equilibrium variable f, factor the expression when appropriate, and then check its sign, for $k = 1, 2, 3$. We illustrate the procedure for $\Delta_1 f(g)$ only, which involves incrementing t_{A}. We consider cases which satisfy $t_{\text{A}} \leqslant 2t_{\text{B}} + 2t_{\text{AB}} + 2$ and $t_{\text{A}} + 1 \leqslant 2t_{\text{B}} + 2t_{\text{AB}} + 2$ so that Proposition C.5(a) is applicable before and after incrementing t_{A}. We also elaborate on $\Delta_3 f(g)$ for $f \in \{q^*_{\text{A}}, m^*_{\text{A}}, \pi^*_{\text{A}}\}$ when $t_{\text{A}} \leqslant 2t_{\text{B}} + 2t_{\text{AB}} + 2$ that results in the peculiar case of the cell with "+/−" in column (III) of Table 1 in the proof of Proposition C.6 that follows.

With some algebraic computation we find

$$\Delta_1 q^*_{\text{C}}(g) = \frac{(\alpha - v_T)(t_{\text{B}} + 2t_{\text{AB}} + 2)^2}{\beta L(t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}) L(t_{\text{A}}+1, t_{\text{B}}, t_{\text{AB}})} > 0 \tag{C.41}$$

since $\alpha \geqslant v_T$ and $L(x_1,x_2,x_3)>0, \forall x_1,x_2,x_3 \geqslant 0$. Since q^*_{C}, m^*_{C}, and π^*_{C} always change in the same direction, opposite to the change in $p^*_{(1)}$, we have the results for $m^*_{\text{C}}, \pi^*_{\text{C}}$, and $p^*_{(1)}$ as well.

$$\Delta_1\, q^*_{\text{A}}(g) = \frac{2(\alpha - v_T)(t_{\text{AB}}+t_{\text{B}}+1)(2t_{\text{AB}}+t_{\text{B}}+2)}{\beta L(t_{\text{A}},t_{\text{B}},t_{\text{AB}})L(t_{\text{A}}+1,t_{\text{B}},t_{\text{AB}})} > 0. \tag{C.42}$$

This gives the results for $q^*_{\text{A}}, m^*_{\text{A}}$, and π^*_{A}. Similarly

$$\Delta_1\, q^*_{\text{B}}(g) = -\frac{(\alpha - v_T)t_{\text{B}}(2t_{\text{AB}}+t_{\text{B}}+2)}{\beta L(t_{\text{A}},t_{\text{B}},t_{\text{AB}})L(t_{\text{A}}+1,t_{\text{B}},t_{\text{AB}})} < 0 \tag{C.43}$$

This gives the results for $q^*_{\text{B}}, m^*_{\text{B}}$, and π^*_{B}.

$$\Delta_1\, p^*_{(2)\text{A}}(g) = -\frac{2(\alpha - v_T)(2t_{\text{AB}}+t_{\text{B}}+2)(4t_{\text{AB}}+3t_{\text{B}}+4)}{L(t_{\text{A}},t_{\text{B}},t_{\text{AB}})L(t_{\text{A}}+1,t_{\text{B}},t_{\text{AB}})} < 0 \tag{C.44}$$

$$\Delta_1\, p^*_{(2)\text{B}}(g) = -\frac{4(\alpha - v_T)(t_{\text{AB}}+1)(2t_{\text{AB}}+t_{\text{B}}+2)}{L(t_{\text{A}},t_{\text{B}},t_{\text{AB}})L(t_{\text{A}}+1,t_{\text{B}},t_{\text{AB}})} < 0. \tag{C.45}$$

Finally,

$$\Delta_1\, \rho^*_{\text{A}}(g) = \frac{3(2t_{\text{AB}}+t_{\text{B}}+2)(2t_{\text{AB}}+3t_{\text{B}})}{2(4t^2_{\text{AB}}+4t_{\text{A}}t_{\text{AB}}+4t_{\text{B}}t_{\text{AB}}+4t_{\text{AB}}+3t_{\text{A}}+3t_{\text{A}}t_{\text{B}}+3t_{\text{B}})(4t^2_{\text{AB}}+4t_{\text{A}}t_{\text{AB}}+4t_{\text{B}}t_{\text{AB}}+8t_{\text{AB}}+3t_{\text{A}}+3t_{\text{A}}t_{\text{B}}+6t_{\text{B}}+3)} \geqslant 0 \tag{C.46}$$

(">0" if $t_{\text{B}}+t_{\text{AB}}>0$) which gives the results for ρ^*_{B} as well. □

Proposition 1 is a simplified version of Proposition C.6:

PROPOSITION C.6. *Given supply network $g \in G$ where $t_{\text{A}} \leqslant 2t_{\text{B}}+2t_{\text{AB}}+2$, adding a shared supplier to g increases equilibrium variables q^*_{A}, m^*_{A}, and π^*_{A} if and only if*

$$t_{\text{A}} < \theta(t_{\text{B}},t_{\text{AB}}) \equiv \frac{\sqrt{33t^2_{\text{B}}+72t_{\text{B}}t_{\text{AB}}+108t_{\text{B}}+48t^2_{\text{AB}}+144t_{\text{AB}}+100}}{4} - t_{\text{AB}} - \frac{t_{\text{B}}}{4} - \frac{3}{2}. \tag{C.47}$$

Proof. As we have shown in the proof of Theorem 2, $q^*_{\text{A}}, m^*_{\text{A}}$, and π^*_{A} always change in the same direction. Hence it suffices to show the effect on q^*_{A}. Using the result from Proposition C.5, we find

$$\Delta_3\, q^*_{\text{A}}(g) = -\frac{2(\alpha - v_T)(4t_{\text{A}}t_{\text{AB}}-4t^2_{\text{AB}}-8t_{\text{B}}t_{\text{AB}}-12t_{\text{AB}}+2t^2_{\text{A}}+t_{\text{B}}t_{\text{A}}+6t_{\text{A}}-4t^2_{\text{B}}-12t_{\text{B}}-8)}{3\beta L(t_{\text{A}},t_{\text{B}},t_{\text{AB}})L(t_{\text{A}},t_{\text{B}},t_{\text{AB}}+1)}. \tag{C.48}$$

Since $L(x_1,x_2,x_3)>0, \forall x_1,x_2,x_3 \geqslant 0$,

$$\text{sgn}(\Delta_3\, q^*_{\text{A}}(g)) = \text{sgn}(-(4t_{\text{A}}t_{\text{AB}}-4t^2_{\text{AB}}-8t_{\text{B}}t_{\text{AB}}-12t_{\text{AB}}+2t^2_{\text{A}}+t_{\text{B}}t_{\text{A}}+6t_{\text{A}}-4t^2_{\text{B}}-12t_{\text{B}}-8)) \tag{C.49}$$

Note that what is inside the sgn operator on the right-hand side of (C.49) is quadratic in t_{A} with coefficient -2 on t^2_{A} and two roots in $\mathbb{R}$ as follows:

$$t_{\text{A}-} = \frac{1}{4}(-6-t_{\text{B}}-4t_{\text{AB}}-\sqrt{72t_{\text{B}}t_{\text{AB}}+48t^2_{\text{AB}}+144t_{\text{AB}}+33t^2_{\text{B}}+108t_{\text{B}}+100}) \tag{C.50}$$

$$t_{\text{A}+} = \frac{1}{4}(-6-t_{\text{B}}-4t_{\text{AB}}+\sqrt{72t_{\text{B}}t_{\text{AB}}+48t^2_{\text{AB}}+144t_{\text{AB}}+33t^2_{\text{B}}+108t_{\text{B}}+100}) \tag{C.51}$$

It is clear that $t_{\text{A}-}<0$. We next show that $0<t_{\text{A}+}<2t_{\text{B}}+2t_{\text{AB}}+2$. Note

$$(\sqrt{72t_{\text{B}}t_{\text{AB}}+48t^2_{\text{AB}}+144t_{\text{AB}}+33t^2_{\text{B}}+108t_{\text{B}}+100})^2-(-6-t_{\text{B}}-4t_{\text{AB}})^2 = 32(t_{\text{B}}(2t_{\text{AB}}+3)+t^2_{\text{AB}}+3t_{\text{AB}}+t^2_{\text{B}}+2)>0 \tag{C.52}$$

so

$$\sqrt{72t_{\text{B}}t_{\text{AB}}+48t^2_{\text{AB}}+144t_{\text{AB}}+33t^2_{\text{B}}+108t_{\text{B}}+100} > |-6-t_{\text{B}}-4t_{\text{AB}}| \tag{C.53}$$

which is equivalent to $t_{A+} > 0$. On the other hand,

$$(2t_B + 2t_{AB} + 2) - t_{A+} = \frac{1}{4}\left(12t_{AB} + 9t_B + 14 - \sqrt{36t_B(2t_{AB}+3) + 4(12t_{AB}^2 + 36t_{AB} + 25) + 33t_B^2}\right). \tag{C.54}$$

Now

$$(12t_{AB} + 9t_B + 14)^2 - (\sqrt{36t_B(2t_{AB}+3) + 4(12t_{AB}^2 + 36t_{AB} + 25) + 33t_B^2})^2$$
$$= 8[37 + 6t_B^2 + 60t_{AB} + 24t_{AB}^2 + 9t_B(5 + 4t_{AB})] > 0 \tag{C.55}$$

which implies (C.54) is negative. Therefore, when $t_A \leqslant 2t_B + 2t_{AB} + 2$, $\Delta_3(q_A^*) > 0$ if $t_A < t_{A+}$ and $\Delta_3(q_A^*) < 0$ if $t_A > t_{A+}$. □

Proof of Proposition 2. Similar to the proof of Theorem 2; by calculating, factoring, and observing the sign of the relevant difference. □

Appendix D: Proofs for the Auditing Phase

We define $V : \Xi \times \Gamma \to \mathbb{R}$ as the value function. Let $\widetilde{V}(\xi, \gamma, x)$ be expected value of choosing $x \in X_\gamma$ when in state $\gamma \in \Gamma$ and following policy $\xi \in \Xi$ thereon. Therefore, given auditing policy $\xi \in \Xi$ and state $\gamma \in \Gamma$, $V(\xi, \gamma) = \widetilde{V}(\xi, \gamma, \xi(\gamma))$, and

$$\widetilde{V}(\xi, \gamma, \text{PP}) = \pi(\gamma) - \zeta(\gamma) \tag{D.1}$$

and given $i \in U_\gamma$,

$$\widetilde{V}(\xi, \gamma, \text{AD}(i)) = -a + (1-u)V(\xi, \gamma \oplus i) + uV(\xi, \gamma \ominus i). \tag{D.2}$$

$$\widetilde{V}(\xi, \gamma, \text{AR}(i)) = -a + (1-u)V(\xi, \gamma \oplus i) + u(-r + V(\xi, \gamma \oplus i)) \tag{D.3}$$

$$= -a - ur + V(\xi, \gamma \oplus i). \tag{D.4}$$

Recall from Section 5.1 that RP is a shorthand for "audit and rectify (AR) all remaining unaudited suppliers if $a + ur < uwz$ and proceed to production (PP) otherwise" and $c_{RP} \equiv (uwz) \wedge (a + ur)$ is the cost associated with each unaudited supplier in the RP subphase. For any $\gamma \in \Gamma$ and $\xi \in \Xi$ we write $\widetilde{V}^*(\gamma, \text{RP}) = \widetilde{V}(\xi, \gamma, \text{RP}) = \pi(\gamma) - c_{RP}|U_\gamma|$.

Given state $\gamma \in \Gamma$, let $R^+(\gamma) \subseteq \Gamma$ be the set of states reachable from γ (including γ itself): $\gamma' \in R^+(\gamma)$ if and only if there exists a policy $\xi \in \Xi$ such that γ' is reached from γ with strictly positive probability by following ξ.

D.1. Two Subphases of Auditing

Proposition D.1. *The buyer can be at least as well off by postponing all audit and rectify (*AR*) actions to after all audit and drop (*AD*) actions.*

Proof. Let $\xi \in \Xi$ be such that there exists $\gamma = (g, U) \in \Gamma$, $i \in U_\gamma$, and $j \in U_{\gamma \oplus i}$ such that

$$\xi(\gamma) = \text{AR}(i) \quad \text{and} \quad \xi(\gamma \oplus i) = \text{AD}(j). \tag{D.5}$$

(If there does not exist such a triple of γ, i, and j then in ξ already all AR actions come after all AD actions.) We specify a policy $\xi' \in \Xi$ otherwise identical to ξ but with the sequence of the above two actions swapped, namely,

$$\xi'(\gamma') = \xi(\gamma'), \quad \forall \gamma' \in \Gamma \backslash \{\gamma, \gamma \oplus j, \gamma \ominus j\} \tag{D.6}$$

$$\xi'(\gamma) = \text{AD}(j) \tag{D.7}$$

$$\xi'(\gamma \oplus j) = \text{AR}(i) \tag{D.8}$$

$$\xi'(\gamma \ominus j) = \begin{cases} \text{AR}(i), & \text{if } i \notin D_g(j) \\ \xi(\gamma \ominus j), & \text{if } i \in D_g(j) \end{cases}. \tag{D.9}$$

It suffices to show $V(\xi', \gamma) \geqslant V(\xi, \gamma)$.

Now

$$V(\xi, \gamma) = \widetilde{V}(\xi, \gamma, \text{AR}(i)) \tag{D.10}$$
$$= -a - ur + V(\xi, \gamma \oplus i) \tag{D.11}$$
$$= -a - ur + \widetilde{V}(\xi, \gamma \oplus i, \text{AD}(j)) \quad \text{(by (D.5))} \tag{D.12}$$
$$= -a - ur - a + (1-u)V(\xi, \gamma \oplus i \oplus j) + uV(\xi, \gamma \oplus i \ominus j) \quad \text{(by (D.2))} \tag{D.13}$$

and

$$V(\xi', \gamma) = \widetilde{V}(\xi', \gamma, \text{AD}(j)) \tag{D.14}$$
$$= -a + (1-u)V(\xi', \gamma \oplus j) + uV(\xi', \gamma \ominus j). \tag{D.15}$$

There are two cases of i:

- *Case 1:* $i \notin D_g(j)$. Then

$$V(\xi', \gamma) = -a + (1-u)\widetilde{V}(\xi', \gamma \oplus j, \text{AR}(i)) + u\widetilde{V}(\xi', \gamma \ominus j, \text{AR}(i)) \tag{D.16}$$
$$= -a + (1-u)(-a - ur + V(\xi', \gamma \oplus j \oplus i))$$
$$+ u(-a - ur + V(\xi', \gamma \ominus j \oplus i)) \tag{D.17}$$
$$= -a - a - ur + (1-u)V(\xi', \gamma \oplus j \oplus i) + uV(\xi', \gamma \ominus j \oplus i) \tag{D.18}$$

Note that $\xi'|_{R^+(\gamma \oplus j \oplus i)} = \xi|_{R^+(\gamma \oplus i \oplus j)}$, so $V(\xi', \gamma \oplus j \oplus i) = V(\xi, \gamma \oplus i \oplus j)$. Since $i \notin D_g(j)$, $\gamma \ominus j \oplus i = \gamma \oplus i \ominus j$. Also, $\xi'|_{R^+(\gamma \ominus j \oplus i)} = \xi|_{R^+(\gamma \oplus i \ominus j)}$. Hence, $V(\xi', \gamma \ominus j \oplus i) = V(\xi, \gamma \oplus i \ominus j)$. Therefore by comparing (D.13) and (D.18) we conclude $V(\xi', \gamma) = V(\xi, \gamma)$.

- *Case 2:* $i \in D_g(j)$. Immediately, $\gamma \oplus i \ominus j = \gamma \ominus j$.

$$V(\xi', \gamma) = -a + (1-u)\widetilde{V}(\xi', \gamma \oplus j, \text{AR}(i)) + uV(\xi', \gamma \ominus j) \tag{D.19}$$
$$= -a + (1-u)(-a - ur + V(\xi', \gamma \oplus j \oplus i)) + uV(\xi', \gamma \ominus j) \tag{D.20}$$
$$= -a - (1-u)(a + ur) + (1-u)V(\xi', \gamma \oplus j \oplus i) + uV(\xi', \gamma \ominus j). \tag{D.21}$$

Same as above, since $\xi'|_{R^+(\gamma \oplus j \oplus i)} = \xi|_{R^+(\gamma \oplus i \oplus j)}$,

$$V(\xi', \gamma \oplus j \oplus i) = V(\xi, \gamma \oplus i \oplus j). \tag{D.22}$$

Since $i \in D_g(j)$, $\xi'(\gamma \ominus j) = \xi(\gamma \ominus j)$. Hence $\xi'|_{R+(\gamma\ominus j)} = \xi|_{R+(\gamma\ominus j)}$. By $\gamma \oplus i \ominus j = \gamma \ominus j$,

$$V(\xi, \gamma \oplus i \ominus j) = V(\xi, \gamma \ominus j) = V(\xi', \gamma \ominus j). \tag{D.23}$$

Substitute (D.22) and (D.23) into (D.21), then subtract (D.13) to obtain

$$V(\xi', \gamma) - V(\xi, \gamma) = a + ur - (1-u)(a+ur) = u(a+ur) \geqslant 0. \tag{D.24}$$

This completes the proof. Note that the buyer is strictly better off by swapping the actions $\textsc{ar}(i)$ and $\textsc{ad}(j)$ (as in ξ') if and only if $i \in D_g(j)$ and $u(a+ur) > 0$. □

Proposition D.2. *Limit the buyer's actions to audit and rectify (* AR*) unaudited suppliers and proceed to production (* PP*). The optimal auditing policy is to audit and rectify all unaudited suppliers in any sequence if $a + ur \leqslant uwz$ and to proceed to production if $a + ur \geqslant uwz$. Furthermore, given $\gamma \in \Gamma$,*

$$V^*(\gamma) = \pi(\gamma) - [(uwz) \wedge (a+ur)]|U_\gamma| = \pi(\gamma) - c_{\text{RP}}|U_\gamma|. \tag{D.25}$$

Proof. Given $\gamma \in \Gamma$ and $i \in U_\gamma$, by (2) and the definition of ζ,

$$\widetilde{V}^*(\gamma, \textsc{pp}) = \pi(\gamma) - uwz|U_\gamma|. \tag{D.26}$$

We prove the result by mathematical induction on the number of unaudited supplier in the state, $|U_\gamma|$. If $|U_\gamma| = 1$, let $i \in U_\gamma$, then $\gamma \oplus i$ is a terminal state. By (5),

$$\widetilde{V}^*(\gamma, \textsc{ar}(i)) = -a - ur + V^*(\gamma \oplus i) = -a - ur + \pi(\gamma \oplus i) = -a - ur + \pi(\gamma) \tag{D.27}$$

since states $\gamma \oplus i$ and γ have the same underlying supply network, which determines the production profit. Note $\widetilde{V}^*(\gamma, \textsc{ar}(i))$ is independent of i. PP is preferred to $\textsc{ar}(i)$ iff $\widetilde{V}^*(\gamma, \textsc{pp}) \geqslant \widetilde{V}^*(\gamma, \textsc{ar}(i))$, or $\pi(\gamma) - uwz \geqslant -a - ur + \pi(\gamma)$, or $a + ur \geqslant uwz$. Hence,

$$V^*(\gamma) = \pi(\gamma) - (uwz) \wedge (a+ur). \tag{D.28}$$

By mathematical induction, suppose if $|U_\gamma| = m$,

$$V^*(\gamma) = \pi(\gamma) - m[(uwz) \wedge (a+ur)]. \tag{D.29}$$

Now if $|U_\gamma| = m+1$, pick arbitrary $i \in U_\gamma$, then $|U_{\gamma\oplus i}| = m$. By (5) and (D.29),

$$\widetilde{V}^*(\gamma, \textsc{ar}(i)) = -a - ur + V^*(\gamma \oplus i) = -a - ur + \pi(\gamma) - m[(uwz) \wedge (a+ur)]. \tag{D.30}$$

PP is preferred to $\textsc{ar}(i)$ iff $\widetilde{V}^*(\gamma, \textsc{pp}) \geqslant \widetilde{V}^*(\gamma, \textsc{ar}(i))$, or $\pi(\gamma) - (m+1)uwz \geqslant -a - ur + \pi(\gamma) - m[(uwz) \wedge (a+ur)]$, or $a + ur + m[(uwz) \wedge (a+ur)] \geqslant (m+1)uwz$, which holds iff $a + ur \geqslant uwz$, as we wanted to show. Finally, to complete the induction step, note

$$V^*(\gamma) = \begin{cases} \widetilde{V}^*(\gamma, \textsc{pp}), & \text{if } a + ur \geqslant uwz \\ \widetilde{V}^*(\gamma, \textsc{ar}(i)), & \text{if } a + ur \leqslant uwz \end{cases} \tag{D.31}$$

$$= \begin{cases} \pi(\gamma) - (m+1)uwz, & \text{if } a + ur \geqslant uwz \\ -a - ur + \pi(\gamma) - m[(uwz) \wedge (a+ur)], & \text{if } a + ur \leqslant uwz \end{cases} \tag{D.32}$$

$$= \pi(\gamma) - (m+1)[(uwz) \wedge (a+ur)]. \quad \square \tag{D.33}$$

Proof of Theorem 3. The result is a direct consequence of Propositions D.1 and D.2. □

Proof of Corollary 1. The result follows (D.25) in Proposition D.2. □

D.2. Optimal Auditing Sequence

We first identify an optimal policy in a general class of supply networks in Theorem D.1, the proof of which serves as the basis for our proof of Theorem D.2, an expanded and technical version of Theorem 4.

ASSUMPTION D.1 **(decreasing differences of production profit)**. *For any* $\gamma \in \Gamma$ *and any* $i, i' \in U_\gamma, i' \notin D_\gamma(i)$,

$$\nabla(\gamma, i') \leqslant \nabla(\gamma \ominus i, i'). \tag{D.34}$$

To state the next assumption, we define a concept of symmetry for suppliers.

DEFINITION D.1. In state $\gamma = (g, U)$ two unaudited suppliers $i, i' \in U$ are *symmetric* (1) if they belong to the same class of tier-2 suppliers S_{A}, S_{B}, or S_{AB}; or, (2) in the case of $|S_{\text{A}}| = |S_{\text{B}}|$ and $|S_{\text{A}} \cap U| = |S_{\text{B}} \cap U|$, (i) if $i \in S_{\text{A}}$ and $i' \in S_{\text{B}}$, or (ii) if $i = \text{A}$ and $i' = \text{B}$.

ASSUMPTION D.2 **(preservation of LVUS)**. *Let* $\gamma \in \Gamma$ *and* i *be an LVUS in* γ. *Let* $i' \in U_\gamma$ *that is not symmetric with* i. *Then* i *is an LVUS in* $\gamma \ominus i'$, *i.e.,*

$$\nabla(\gamma \ominus i', i) \leqslant \nabla(\gamma \ominus i', i''), \quad \forall i'' \in U_{\gamma \ominus i'}. \tag{D.35}$$

Under Assumption D.2, an LVUS remains an LVUS when we remove a nonsymmetric supplier from the supply network.

Given the assumptions we may completely characterize the optimal auditing policy.

THEOREM D.1. *Let* $\gamma_0 = (g, U) \in \Gamma$ *be such that for any* $\gamma \in R^+(\gamma_0)$, *no unaudited supplier in* γ *is a dependent of another unaudited supplier, i.e., any* $i, i' \in U_\gamma$ $(i \neq i')$ *satisfy* $i \notin D_g(i')$ *and* $i' \notin D_g(i)$. *Under Assumptions D.1 and D.2, the following policy* ξ^* *is optimal in every state* $\gamma \in R^+(\gamma_0)$:

$$\xi^*(\gamma) = \begin{cases} \text{AD}(i), & \text{if } i \in U_\gamma, u\nabla(\gamma, i) + a < c_{\text{RP}}, \text{ and } \nabla(\gamma, i) \leqslant \nabla(\gamma, i'), \forall i' \in U_\gamma \\ \text{RP}, & \text{if } u\nabla(\gamma, i) + a \geqslant c_{\text{RP}}, \forall i \in U_\gamma \end{cases}. \tag{D.36}$$

Proof. We prove the result by mathematical induction on the number of unaudited supplier in the state, $|U_\gamma|$. If $|U_\gamma| = 1$, let $i \in U_\gamma$, then $\widetilde{V}^*(\gamma, \text{AD}(i)) > \widetilde{V}^*(\gamma, \text{RP})$ if and only if

$$-a + (1-u)V^*(\gamma \oplus i) + uV^*(\gamma \ominus i) > \pi(\gamma) - c_{\text{RP}} \tag{D.37}$$

if and only if

$$-a + (1-u)\pi(\gamma \oplus i) + u\pi(\gamma \ominus i) > \pi(\gamma) - c_{\text{RP}}. \tag{D.38}$$

But $\pi(\gamma \oplus i) = \pi(\gamma)$, so above is equivalent to

$$-a - u(\pi(\gamma \oplus i) - \pi(\gamma \ominus i)) > -c_{\text{RP}} \tag{D.39}$$

equivalent to the condition stipulated by ξ^* for taking action $\text{AD}(i)$. Therefore ξ^* is optimal at γ.

By way of mathematical induction, suppose ξ^* is optimal for all $\gamma' \in R^+(\gamma_0)$ such that $|U_{\gamma'}| \leqslant k \in \mathbb{N}^+$. Let $\gamma \in R^+(\gamma_0)$ be such that $|U_\gamma| = k+1$. We divide the proof of the induction step into two cases based on (D.36).

Case a ($u\nabla(\gamma,i)+a\geqslant c_{\text{RP}},\forall i\in U_\gamma$). We show that $\widetilde{V}^*(\gamma,\text{RP})\geqslant \widetilde{V}^*(\gamma,\text{AD}(i))$ for any $i\in U_\gamma$, thus proving the optimality of the action RP when $u\nabla(\gamma,i)+a\geqslant c_{\text{RP}},\forall i\in U_\gamma$ as Theorem D.1 prescribes. Let $i\in U_\gamma$. We first show two equalities: $V^*(\gamma\oplus i)=\widetilde{V}^*(\gamma\oplus i,\text{RP})$ and $V^*(\gamma\ominus i)=\widetilde{V}^*(\gamma\ominus i,\text{RP})$.

First consider the state $\gamma\oplus i$. Note $\pi(\gamma\oplus i)=\pi(\gamma)$ and for any $i'\in U_\gamma\backslash\{i\}$, $\pi(\gamma\oplus i\ominus i')=\pi(\gamma\ominus i')$. Then for any $i'\in U_{\gamma\oplus i}=U_\gamma\backslash\{i\}$,

$$u\nabla(\gamma\oplus i,i')+a=u(\pi(\gamma\oplus i)-\pi(\gamma\oplus i\ominus i'))+a=u(\pi(\gamma)-\pi(\gamma\ominus i'))+a=u\nabla(\gamma,i')+a\geqslant c_{\text{RP}}. \tag{D.40}$$

Hence by the definition of ξ^*, $\xi^*(\gamma\oplus i)=\text{RP}$, i.e., ξ^* prescribes the action RP in state $\gamma\oplus i$. But $|U_{\gamma\oplus i}|=k$. By invoking the induction hypothesis (that ξ^* is optimal at any state $\gamma'\in R^+(\gamma_0)$ such that $|U_{\gamma'}|\leqslant k$), we conclude that the action RP is optimal at state $\gamma\oplus i$. Therefore $V^*(\gamma\oplus i)=\widetilde{V}^*(\gamma\oplus i,\text{RP})$.

Next consider the state $\gamma\ominus i$. Since no unaudited supplier in γ is a dependent of another, $U_{\gamma\ominus i}=U_\gamma\backslash\{i\}$. By Assumption D.1, for any $i'\in U_{\gamma\ominus i}$,

$$u(\pi(\gamma\ominus i)-\pi(\gamma\ominus i\ominus i'))+a\geqslant u(\pi(\gamma)-\pi(\gamma\ominus i'))+a=u\nabla(\gamma,i')+a \tag{D.41}$$

which we know is greater than or equal to c_{RP} for any $i'\in U_\gamma$. Hence $\xi^*(\gamma\ominus i)=\text{RP}$. But $|U_{\gamma\ominus i}|\leqslant k$. By invoking the induction hypothesis (that ξ^* is optimal at any state $\gamma'\in R^+(\gamma_0)$ such that $|U_{\gamma'}|\leqslant k$), we conclude that the action RP is optimal in state $\gamma\ominus i$. Therefore $V^*(\gamma\ominus i)=\widetilde{V}^*(\gamma\ominus i,\text{RP})$.

Now

$$\begin{aligned}
\widetilde{V}^*(\gamma,\text{RP})&=\pi(\gamma)-c_{\text{RP}}|U_\gamma| &\text{(D.42)}\\
&=\pi(\gamma)-c_{\text{RP}}(|U_\gamma|-1)-c_{\text{RP}} &\text{(D.43)}\\
&\geqslant \pi(\gamma)-c_{\text{RP}}(|U_\gamma|-1)-[u(\pi(\gamma)-\pi(\gamma\ominus i))+a] &\text{(D.44)}\\
&=-a+(1-u)(\pi(\gamma\oplus i)-c_{\text{RP}}|U_{\gamma\oplus i}|)+u(\pi(\gamma\ominus i)-c_{\text{RP}}|U_{\gamma\ominus i}|) &\text{(D.45)}\\
&=-a+(1-u)\widetilde{V}^*(\gamma\oplus i,\text{RP})+u\widetilde{V}^*(\gamma\ominus i,\text{RP}) &\text{(D.46)}\\
&=-a+(1-u)V^*(\gamma\oplus i)+uV^*(\gamma\ominus i) &\text{(D.47)}\\
&=\widetilde{V}^*(\gamma,\text{AD}(i)) &\text{(D.48)}
\end{aligned}$$

where (D.44) is by the assumption $u(\pi(\gamma)-\pi(\gamma\ominus i))+a\geqslant c_{\text{RP}}$; (D.45) is by $|U_{\gamma\oplus i}|=|U_{\gamma\ominus i}|=|U_\gamma|-1$ (no unaudited supplier in γ is a dependent of another so that $\gamma\ominus i$ has exactly one less unaudited supplier than γ); and (D.47) is by $V^*(\gamma\oplus i)=\widetilde{V}^*(\gamma\oplus i,\text{RP})$ and $V^*(\gamma\ominus i)=\widetilde{V}^*(\gamma\ominus i,\text{RP})$.

Case b ($\exists i'\in U_\gamma$ such that $u\nabla(\gamma,i')+a<c_{\text{RP}}$). Let $i\in U_\gamma$ be an LVUS in γ, i.e., $\nabla(\gamma,i)\leqslant\nabla(\gamma,j),\forall j\in U_\gamma$. We first show that $\widetilde{V}^*(\gamma,\text{AD}(i))\geqslant\widetilde{V}^*(\gamma,\text{RP})$, then show that $\widetilde{V}^*(\gamma,\text{AD}(i))\geqslant\widetilde{V}^*(\gamma,\text{AD}(i'))$ for any $i'\in U_\gamma$. With these we prove that if i is an LVUS in γ and $u\nabla(\pi,i)+a<c_{\text{RP}}$ then the optimal action to take in state γ is $\text{AD}(i)$ as Theorem D.1 prescribes. Now

$$\begin{aligned}
\widetilde{V}^*(\gamma,\text{AD}(i))&=-a+(1-u)V^*(\gamma\oplus i)+uV^*(\gamma\ominus i) &\text{(D.49)}\\
&\geqslant -a+(1-u)\widetilde{V}^*(\gamma\oplus i,\text{RP})+u\widetilde{V}^*(\gamma\ominus i,\text{RP}) &\text{(D.50)}\\
&=-a+(1-u)(\pi(\gamma\oplus i)-c_{\text{RP}}|U_{\gamma\oplus i}|)+u(\pi(\gamma\ominus i)-c_{\text{RP}}|U_{\gamma\ominus i}|) &\text{(D.51)}\\
&=-a+\pi(\gamma)-c_{\text{RP}}|U_{\gamma\oplus i}|-u(\pi(\gamma)-\pi(\gamma\ominus i)) &\text{(D.52)}\\
&>\pi(\gamma)-c_{\text{RP}}|U_\gamma| &\text{(D.53)}\\
&=\widetilde{V}^*(\gamma,\text{RP}) &\text{(D.54)}
\end{aligned}$$

where (D.50) is by V^* being optimal; (D.52) is by $\pi(\gamma\oplus i)=\pi(\gamma)$; and (D.53) is by $|U_{\gamma\oplus i}|=|U_\gamma|-1$ and the premise of case b.

We next show that $\widetilde{V}^*(\gamma,\text{AD}(i))\geqslant \widetilde{V}^*(\gamma,\text{AD}(i'))$ for any $i'\in U_\gamma$. Let $i'\in U_\gamma$ such that i' is not symmetric with i. (If i' is symmetric with i, clearly $\widetilde{V}^*(\gamma,\text{AD}(i))=\widetilde{V}^*(\gamma,\text{AD}(i'))$.) Since i is an LVUS in $\gamma\oplus i'$ and $u(\pi(\gamma\oplus i')-\pi(\gamma\oplus i'\ominus i))+a=u(\pi(\gamma)-\pi(\gamma\ominus i))+a<c_{\text{RP}}$ (by the premise of case b), by the induction hypothesis, $\xi^*(\gamma\oplus i')=\text{AD}(i)$. On the other hand, by Assumption D.2, i is an LVUS in $\gamma\ominus i'$; therefore[11]

$$\xi^*(\gamma\ominus i')=\begin{cases}\text{AD}(i), & \text{if } u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))+a<c_{\text{RP}}\\ \text{AR}(i), & \text{if } u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))+a\geqslant c_{\text{RP}} \text{ and } a+ur<uwz\\ \text{PP}, & \text{if } u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))+a\geqslant c_{\text{RP}} \text{ and } a+ur\geqslant uwz\end{cases}. \tag{D.55}$$

We next look at the three cases in (D.55) separately. In each case we devise a policy $\widehat{\xi}$ so that the buyer's expected profit from first taking the action AD(i) and following $\widehat{\xi}$ thereafter is at least as good as the expected profit from first taking AD(i') and following the optimal policy ξ^* thereafter (ξ^* is optimal thereafter by the induction hypothesis). That is, $\widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i))\geqslant\widetilde{V}(\xi^*,\gamma,\text{AD}(i'))$. Since $\widetilde{V}^*(\gamma,\text{AD}(i))\geqslant\widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i)))$ and $\widetilde{V}(\xi^*,\gamma,\text{AD}(i'))=\widetilde{V}^*(\gamma,\text{AD}(i'))$, we must then have $\widetilde{V}^*(\gamma,\text{AD}(i))\geqslant\widetilde{V}^*(\gamma,\text{AD}(i'))$ as desired. In each case we consider the following four events that together form a partition of the sample space:

$$H_{11}=\{\text{both } i \text{ and } i' \text{ are compliant}\} \tag{D.56}$$

$$H_{10}=\{i \text{ is compliant and } i' \text{ is not compliant}\} \tag{D.57}$$

$$H_{01}=\{i \text{ is not compliant and } i' \text{ is compliant}\} \tag{D.58}$$

$$H_{00}=\{\text{neither } i \text{ nor } i' \text{ is compliant}\}. \tag{D.59}$$

Case b(i) $(u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))+a<c_{\text{RP}})$. Let $\widehat{\xi}\in\Xi$ be the policy such that $\widehat{\xi}(\gamma\oplus i)=\widehat{\xi}(\gamma\ominus i)=$ AD(i') and $\widehat{\xi}(\gamma')=\xi^*(\gamma')$ for any $\gamma'\in\Gamma\backslash\{\gamma\oplus i,\gamma\ominus i\}$.

Conditional on H_{11}: The path of state transition by taking AD(i) at state γ then following policy $\widehat{\xi}$ is

$$\gamma\xrightarrow{\text{AD}(i)}\gamma\oplus i\xrightarrow{\text{AD}(i')}\gamma\oplus i\oplus i' \tag{D.60}$$

while that by taking AD(i') at state γ then following policy ξ^* is

$$\gamma\xrightarrow{\text{AD}(i')}\gamma\oplus i'\xrightarrow{\text{AD}(i)}\gamma\oplus i'\oplus i. \tag{D.61}$$

Note that $\gamma\oplus i\oplus i'=\gamma\oplus i'\oplus i$ and $\widehat{\xi}|_{R^+(\gamma\oplus i\oplus i')}=\xi^*|_{R^+(\gamma\oplus i'\oplus i)}$, so the expected profit at γ from first taking AD(i) then following policy $\widehat{\xi}$ is the same as that from first taking AD(i') then following ξ^* conditional on H_{11}.

Conditional on H_{10}: The path of state transition by taking AD(i) at state γ then following policy $\widehat{\xi}$ is

$$\gamma\xrightarrow{\text{AD}(i)}\gamma\oplus i\xrightarrow{\text{AD}(i')}\gamma\oplus i\ominus i' \tag{D.62}$$

while that by taking AD(i') at state γ then following policy ξ^* is

$$\gamma\xrightarrow{\text{AD}(i')}\gamma\ominus i'\xrightarrow{\text{AD}(i)}\gamma\ominus i'\oplus i. \tag{D.63}$$

[11] If $u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))+a\geqslant c_{\text{RP}}$ and $a+ur<uwz$, ξ^* prescribes auditing and rectify (if noncompliant) all unaudited suppliers in any sequence; here we choose i to audit next.

Note that $\gamma \oplus i \ominus i' = \gamma \ominus i' \oplus i$ and $\widehat{\xi}|_{R^+(\gamma \oplus i \ominus i')} = \xi^*|_{R^+(\gamma \ominus i' \oplus i)}$, so the expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{10}.

Conditional on H_{01} *or* H_{00}: Similarly we can show that the expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^*.

Therefore the unconditional expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ and that from first taking $\text{AD}(i')$ then following ξ^*, which are integrals of the respected conditional expected profits, must be equal; that is $\widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)) = \widetilde{V}(\xi^*, \gamma, \text{AD}(i'))$. Therefore

$$\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)) = \widetilde{V}(\xi^*, \gamma, \text{AD}(i')) = \widetilde{V}^*(\gamma, \text{AD}(i')) \tag{D.64}$$

where the induction hypothesis (that ξ^* is optimal at any state $\gamma' \in R^+(\gamma_0)$ with $|U_{\gamma'}| \leqslant k$) gives the last equality.

Case b(ii) $(u(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i)) + a \geqslant c_{\text{RP}}$ and $a + ur < uwz)$. Let $\widehat{\xi} \in \Xi$ be the policy such that $\widehat{\xi}(\gamma \oplus i) = \text{AD}(i')$, $\widehat{\xi}(\gamma \ominus i) = \text{AR}(i')$, and $\widehat{\xi}(\gamma') = \xi^*(\gamma')$ for any $\gamma' \in \Gamma \backslash \{\gamma \oplus i, \gamma \ominus i\}$.

Conditional on H_{11}: Using the same steps as in case b(i) we can show the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{11}.

Conditional on H_{10}: The path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \oplus i \xrightarrow{\text{AD}(i')} \gamma \oplus i \ominus i' \tag{D.65}$$

while that by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{AR}(i)} \gamma \ominus i' \oplus i. \tag{D.66}$$

Note that $\gamma \oplus i \ominus i' = \gamma \ominus i' \oplus i$ and $\widehat{\xi}|_{R^+(\gamma \oplus i \ominus i')} = \xi^*|_{R^+(\gamma \ominus i' \oplus i)}$, so the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{10}.

Conditional on H_{01}: Similarly we can show that the expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^*.

Conditional on H_{00}: The path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \ominus i \xrightarrow{\text{AR}(i')} \gamma \ominus i \oplus i' \tag{D.67}$$

while that by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{AR}(i)} \gamma \ominus i' \oplus i. \tag{D.68}$$

Since $u(\pi(\gamma \ominus i' \oplus i) - \pi(\gamma \ominus i' \oplus i \ominus i'')) + a = u(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i'')) + a \geqslant u(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i)) + a$ for any $i'' \in U_{\gamma \ominus i' \oplus i}$ (the last inequality is because i is an LVUS in $\gamma \ominus i'$, by Assumption D.2), and $u(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i)) + a \geqslant c_{\text{RP}}$ (premise of case b(ii)), we have $u(\pi(\gamma \ominus i' \oplus i) - \pi(\gamma \ominus i' \oplus i \ominus i'')) + a \geqslant c_{\text{RP}}$. Therefore $\xi^*(\gamma \ominus i' \oplus i) = \text{RP}$. Note that since $\widehat{\xi}|_{R^+(\gamma \ominus i \oplus i')} = \xi^*|_{R^+(\gamma \ominus i \oplus i')}$,

$$V(\widehat{\xi}, \gamma \ominus i \oplus i') = V^*(\gamma \ominus i \oplus i') \geqslant \widetilde{V}^*(\gamma \ominus i \oplus i', \text{RP}). \tag{D.69}$$

On the other hand, since $|U_{\gamma\ominus i\oplus i'}|=|U_{\gamma\ominus i'\oplus i}|$,

$$\widetilde{V}^*(\gamma\ominus i\oplus i',\text{RP})-\widetilde{V}^*(\gamma\ominus i'\oplus i,\text{RP})=\pi(\gamma\ominus i\oplus i')-\pi(\gamma\ominus i'\oplus i)=\pi(\gamma\ominus i)-\pi(\gamma\ominus i')\geqslant 0 \tag{D.70}$$

Together they imply

$$V(\widehat{\xi},\gamma\ominus i\oplus i')\geqslant\widetilde{V}^*(\gamma\ominus i'\oplus i,\text{RP})=V^*(\gamma\ominus i'\oplus i) \tag{D.71}$$

where the last equality is because $\xi^*(\gamma\ominus i'\oplus i)=\text{RP}$. Therefore the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is greater than or equal to that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{10}.

Therefore the unconditional expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is greater than or equal to that from first taking $\text{AD}(i')$ then following ξ^*; that is $\widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i))\geqslant\widetilde{V}(\xi^*,\gamma,\text{AD}(i'))$. Therefore

$$\widetilde{V}^*(\gamma,\text{AD}(i))\geqslant\widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i))\geqslant\widetilde{V}(\xi^*,\gamma,\text{AD}(i'))=\widetilde{V}^*(\gamma,\text{AD}(i')) \tag{D.72}$$

where the induction hypothesis gives the last equality.

Case b(iii) $(u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))+a\geqslant c_{\text{RP}}$ and $a+ur\geqslant uwz)$. Let $\widehat{\xi}\in\Xi$ be the policy such that (1) $\widehat{\xi}(\gamma\oplus i)=\text{AD}(i')$, (2) for any $\gamma'\in R^+(\gamma\ominus i)$ such that $i'\in U_{\gamma'}$, $\widehat{\xi}(\gamma')=\xi^*(\gamma'\ominus i')$, and (3) $\widehat{\xi}(\gamma')=\xi^*(\gamma')$ for any other state γ' (i.e., $\gamma'\in\Gamma\backslash\{\gamma\oplus i\}\backslash\{\gamma''\in R^+(\gamma\ominus i):i'\in U_{\gamma''}\}$).

Conditional on H_{11}: Using the same corresponding steps as in case b(i) we can show the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{11}.

Conditional on H_{10}: Since i is an LVUS in $\gamma\ominus i'$, by the premise of case b(iii), any unaudited supplier i'' in state $\gamma\oplus i\ominus i'$ must have $u\nabla(\gamma\oplus i\ominus i',i'')+a=u\nabla(\gamma\ominus i',i'')+a\geqslant c_{\text{RP}}$, then the induction hypothesis implies $\xi^*(\gamma\oplus i\ominus i')=\text{PP}$. By the definition of $\widehat{\xi}$, $\widehat{\xi}(\gamma\oplus i\ominus i')=\xi^*(\gamma\oplus i\ominus i')$. Therefore $\widehat{\xi}(\gamma\oplus i\ominus i')=\text{PP}$. Then the path of the state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma\xrightarrow{\text{AD}(i)}\gamma\oplus i\xrightarrow{\text{AD}(i')}\gamma\oplus i\ominus i'\xrightarrow{\text{PP}}. \tag{D.73}$$

The path of the state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma\xrightarrow{\text{AD}(i')}\gamma\ominus i'\xrightarrow{\text{PP}}. \tag{D.74}$$

Note that $\pi(\gamma\oplus i\ominus i')=\pi(\gamma\ominus i')$, so the only difference in the conditional expected profit between the above two paths is the additional cost a of carrying out one more audit in (D.73) (since i is compliant on H_{10} it will not incur any penalty from violation later on).

Conditional on H_{01}: The path of the state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma\xrightarrow{\text{AD}(i)}\gamma\ominus i \tag{D.75}$$

while the path of the state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma\xrightarrow{\text{AD}(i')}\gamma\oplus i'\xrightarrow{\text{AD}(i)}\gamma\oplus i'\ominus i. \tag{D.76}$$

Note that the definition of $\widehat{\xi}$ means that the path subsequent to $\gamma\ominus i$ in (D.75) and that subsequent to $\gamma\oplus i'\ominus i$ in (D.76) will be identical except that i' will remain unaudited in all subsequent states in (D.75) while it is vetted in (D.76). Since on H_{01} i' is compliant the only difference in the conditional expected profit between the above two paths is the additional cost a of carrying out one more audit in (D.76) (since i' is compliant on H_{10}, even if unaudited, it will not incur any penalty from violation later on).

Conditional on H_{00}: The path of the state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \ominus i \xrightarrow{\text{PP}} \tag{D.77}$$

where $\widehat{\xi}(\gamma \ominus i) = \xi^*(\gamma \ominus i \ominus i') = \text{PP}$ by the premise of the current case and Assumption D.1 (so that $u(\pi(\gamma \ominus i \ominus i') - \pi(\gamma \ominus i \ominus i' \ominus i'')) + a \geqslant c_{\text{RP}}, \forall i'' \in U_{\gamma \ominus i \ominus i'}$), while the path of the state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{PP}} . \tag{D.78}$$

Therefore conditional on H_{00} the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is greater than that from first taking $\text{AD}(i')$ then following ξ^* by precisely $\pi(\gamma \ominus i) - \pi(\gamma \ominus i') \geqslant 0$.

Therefore the unconditional expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is greater than or equal to that from first taking $\text{AD}(i')$ then following ξ^*; that is $\widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi^*, \gamma, \text{AD}(i'))$. Therefore

$$\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi^*, \gamma, \text{AD}(i')) = \widetilde{V}^*(\gamma, \text{AD}(i')) \tag{D.79}$$

where the induction hypothesis yields the last equality.

To sum up, in all cases b(i)–b(iii), $\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}^*(\gamma, \text{AD}(i'))$.

ξ^* is optimal at γ. □

Theorem 4 is a shortened and less technical version of Theorem D.2.

THEOREM D.2. *Under Condition 1 the following policy ξ^{**} is optimal at any state γ in which every tier-1 firm is vetted: for any nonterminal state $\gamma \neq \gamma_1$, let i be an LVUS in γ, then*

$$\xi^{**}(\gamma) = \begin{cases} \text{AD}(i), & \text{if } u\nabla(\gamma, i) + a < c_{\text{RP}} \\ \text{RP}, & \text{if } u\nabla(\gamma, i) + a \geqslant c_{\text{RP}} \end{cases} \tag{D.80}$$

and for γ_1 and $i \in U_{\gamma_1}$,

$$\xi^{**}(\gamma_1) = \begin{cases} \text{AD}(i), & \text{if } \dfrac{1}{1+u}(a + u\nabla(\gamma_1, i)) + \dfrac{u}{1+u}(a + u\pi(\gamma_1 \ominus i)) < c_{\text{RP}} \\ \text{RP}, & \text{if } \dfrac{1}{1+u}(a + u\nabla(\gamma_1, i)) + \dfrac{u}{1+u}(a + u\pi(\gamma_1 \ominus i)) \geqslant c_{\text{RP}} \end{cases} . \tag{D.81}$$

The policy ξ^{**} differs from ξ^* only at state γ_1 where Assumption D.1 fails. At state γ_1, ξ^{**} prescribes $\text{AD}(i)$ in a larger region of the parameter space than ξ^* does, since ξ^{**} takes into account the fact that if the buyer drops i, the last remaining unaudited supplier will be even less valuable. The buyer has less incentive to keep the supply network operating in state γ_1 than in states in which decreasing differences hold.

Proof of Theorem D.2. Since we limit to states in which all tier-1 firms are vetted, (1) by Proposition 2 Assumption D.2 holds, and (2) no unaudited supplier can be a dependent of another unaudited supplier. Under Condition 1, among all states we consider here the only state at which Assumption D.1 fails is γ_1, the induction proof of Theorem D.1 applies directly by replacing ξ^* with ξ^{**}, with two exceptions: (1) at γ_1 itself, at which state we show the optimality of ξ^{**} separately, and (2) at state $\gamma_2 = (g, U)$ where $g = (\{\text{A}, \text{B}\}, \{1\}, \{3\}, \{2\})$ and $U = \{1, 2, 3\}$, and if the LVUS, 1, in γ_2 satisfies $u\nabla(\gamma_2, i) + a < c_{\text{RP}}$. We will go on to show why the induction proof still applies in the second case.

ξ^{**} *is optimal at* γ_1 Set $\gamma = \gamma_1$. Call the two symmetric tier-2 suppliers in γ_1 i and i'. To analyze the decision at γ we first consider the profits in state $\gamma \oplus i$ and $\gamma \ominus i$. In state $\gamma \oplus i$ the only unaudited supplier is i'. The decision is between $\text{AD}(i')$ (with expected profit $-a + (1-u)\pi(\gamma \oplus i \oplus i') + u\pi(\gamma \oplus i \ominus i')$) and RP (with expected profit $\pi(\gamma \oplus i) - c_{\text{RP}}$). Therefore

$$V^*(\gamma \oplus i) = \begin{cases} -a + (1-u)\pi(\gamma \oplus i \oplus i') + u\pi(\gamma \oplus i \ominus i'), & \text{if } u(\pi(\gamma \oplus i \oplus i') - \pi(\gamma \oplus i \ominus i')) + a < c_{\text{RP}} \\ \pi(\gamma \oplus i) - c_{\text{RP}}, & \text{if } u(\pi(\gamma \oplus i \oplus i') - \pi(\gamma \oplus i \ominus i')) + a \geqslant c_{\text{RP}} \end{cases} \tag{D.82}$$

$$= \begin{cases} -a + (1-u)\pi(\gamma) + u\pi(\gamma \ominus i'), & \text{if } u(\pi(\gamma) - \pi(\gamma \ominus i')) + a < c_{\text{RP}} \\ \pi(\gamma) - c_{\text{RP}}, & \text{if } u(\pi(\gamma) - \pi(\gamma \ominus i')) + a \geqslant c_{\text{RP}} \end{cases}. \tag{D.83}$$

Similary, in state $\gamma \ominus i$ the only unaudited supplier is i'. The decision is between $\text{AD}(i')$ (with expected profit $-a + (1-u)\pi(\gamma \ominus i \oplus i')$) and RP (with expected profit $\pi(\gamma \ominus i) - c_{\text{RP}}$). Therefore

$$V^*(\gamma \ominus i) = \begin{cases} -a + (1-u)\pi(\gamma \ominus i \oplus i'), & \text{if } u\pi(\gamma \ominus i \oplus i') + a < c_{\text{RP}} \\ \pi(\gamma \ominus i) - c_{\text{RP}} & \text{if } u\pi(\gamma \ominus i \oplus i') + a \geqslant c_{\text{RP}} \end{cases} \tag{D.84}$$

$$= \begin{cases} -a + (1-u)\pi(\gamma \ominus i), & \text{if } u\pi(\gamma \ominus i) + a < c_{\text{RP}} \\ \pi(\gamma \ominus i) - c_{\text{RP}} & \text{if } u\pi(\gamma \ominus i) + a \geqslant c_{\text{RP}} \end{cases}. \tag{D.85}$$

By Proposition C.5 we algebraically verify that

$$\pi(\gamma) - \pi(\gamma \ominus i') > \pi(\gamma \ominus i') = \pi(\gamma \ominus i) \tag{D.86}$$

(which is how Assumption D.1 is violated). By (D.83) and (D.85) we obtain

$$\widetilde{V}^*(\gamma, \text{AD}(i)) = -a + (1-u)V^*(\gamma \oplus i) + uV^*(\gamma \ominus i) \tag{D.87}$$

$$= \begin{cases} -2a + (1-u)[(1-u)\pi(\gamma) + u\pi(\gamma \ominus i')] + u[(1-u)\pi(\gamma \ominus i)], & \text{if } u(\pi(\gamma) - \pi(\gamma \ominus i')) + a < c_{\text{RP}} \\ -a + (1-u)[\pi(\gamma) - c_{\text{RP}}] + u[-a + (1-u)\pi(\gamma \ominus i)], & \text{if } \begin{array}{l} u\pi(\gamma \ominus i) + a < c_{\text{RP}} \\ \leqslant u(\pi(\gamma) - \pi(\gamma \ominus i')) + a \end{array} \\ -a + (1-u)[\pi(\gamma) - c_{\text{RP}}] + u[\pi(\gamma \ominus i) - c_{\text{RP}}], & \text{if } u\pi(\gamma \ominus i) + a \geqslant c_{\text{RP}} \end{cases} \tag{D.88}$$

$$= \begin{cases} -2a + (1-u)^2\pi(\gamma) + 2u(1-u)\pi(\gamma \ominus i'), & \text{if } u(\pi(\gamma) - \pi(\gamma \ominus i')) + a < c_{\text{RP}} \\ -a + (1-u)[\pi(\gamma) - c_{\text{RP}}] + u[-a + (1-u)\pi(\gamma \ominus i)], & \text{if } \begin{array}{l} u\pi(\gamma \ominus i) + a < c_{\text{RP}} \\ \leqslant u(\pi(\gamma) - \pi(\gamma \ominus i')) + a \end{array} \\ -a - c_{\text{RP}} + (1-u)\pi(\gamma) + u\pi(\gamma \ominus i), & \text{if } u\pi(\gamma \ominus i) + a \geqslant c_{\text{RP}} \end{cases}. \tag{D.89}$$

On the other hand $\widetilde{V}^*(\gamma, \text{RP}) = \pi(\gamma) - c_{\text{RP}}|U_\gamma| = \pi(\gamma) - 2c_{\text{RP}}$. Hence $\widetilde{V}^*(\gamma, \text{AD}(i)) > \widetilde{V}^*(\gamma, \text{RP})$ if and only if one of the following three (mutually exclusive) conditions holds:

(*a*) $a + u(\pi(\gamma) - \pi(\gamma \ominus i')) < c_{\text{RP}}$ and $-2a + (1-u)^2\pi(\gamma) + 2u(1-u)\pi(\gamma \ominus i') > \pi(\gamma) - 2c_{\text{RP}}$;

(*b*) $a + u\pi(\gamma \ominus i) < c_{\text{RP}} \leqslant a + u(\pi(\gamma) - \pi(\gamma \ominus i'))$ and $-a + (1-u)[\pi(\gamma) - c_{\text{RP}}] + u[-a + (1-u)\pi(\gamma \ominus i)] > \pi(\gamma) - 2c_{\text{RP}}$;

(*c*) $a + u\pi(\gamma \ominus i) \geqslant c_{\text{RP}}$ and $-a - c_{\text{RP}} + (1-u)\pi(\gamma) + u\pi(\gamma \ominus i) > \pi(\gamma) - 2c_{\text{RP}}$.

In (*a*), the second inequality is equivalent to

$$2[a + u(\pi(\gamma) - \pi(\gamma \ominus i'))] - u^2[(\pi(\gamma) - \pi(\gamma \ominus i')) - \pi(\gamma \ominus i')] < 2c_{\text{RP}} \tag{D.90}$$

which is implied by the first inequality and that $(\pi(\gamma) - \pi(\gamma \ominus i')) > \pi(\gamma \ominus i')$ which we know to be true. So (*a*) can be simplified to just $a + u(\pi(\gamma) - \pi(\gamma \ominus i')) < c_{\text{RP}}$. In (*b*), the last inequality is equivalent to

$$[a + u(\pi(\gamma) - \pi(\gamma \ominus i))] + u(a + u\pi(\gamma \ominus i)) < (1+u)c_{\text{RP}} \tag{D.91}$$

or

$$\frac{1}{1+u}[a+u(\pi(\gamma)-\pi(\gamma\ominus i))]+\frac{u}{1+u}(a+u\pi(\gamma\ominus i))<c_{\text{RP}}. \tag{D.92}$$

Note that (D.92) and the second inequality $c_{\text{RP}}\leqslant a+u(\pi(\gamma)-\pi(\gamma\ominus i'))$ implies the first inequality $a+u\pi(\gamma\ominus i)<c_{\text{RP}}$. So (b) can be simplified to

$$\frac{1}{1+u}[a+u(\pi(\gamma)-\pi(\gamma\ominus i))]+\frac{u}{1+u}(a+u\pi(\gamma\ominus i))<c_{\text{RP}}\leqslant a+u(\pi(\gamma)-\pi(\gamma\ominus i')). \tag{D.93}$$

In (c), the second inequality is equivalent to

$$a+u(\pi(\gamma)-\pi(\gamma\ominus i))<c_{\text{RP}} \tag{D.94}$$

directly contradicting the first inequality; (c) can never hold. Therefore the three conditions above is equivalent to either one of the following two conditions holds:

(a) $a+u(\pi(\gamma)-\pi(\gamma\ominus i'))<c_{\text{RP}}$;

(b) $\frac{1}{1+u}[a+u(\pi(\gamma)-\pi(\gamma\ominus i))]+\frac{u}{1+u}(a+u\pi(\gamma\ominus i))<c_{\text{RP}}\leqslant a+u(\pi(\gamma)-\pi(\gamma\ominus i'))$

which is obviously also equivalent to just

$$\frac{1}{1+u}[a+u(\pi(\gamma)-\pi(\gamma\ominus i))]+\frac{u}{1+u}(a+u\pi(\gamma\ominus i))<c_{\text{RP}} \tag{D.95}$$

since $a+u(\pi(\gamma)-\pi(\gamma\ominus i))>a+u\pi(\gamma\ominus i)$. This shows the optimality of ξ^{**} at γ_1.

ξ^{**} is optimal at γ_2 The only case to show is when the LVUS $i\in U_{\gamma_2}$ of γ_2 satisfies $u(\pi(\gamma_2)-\pi(\gamma_2\ominus i))+a<c_{\text{RP}}$. The proof is analogous to case b in the proof of Theorem D.1 by replacing ξ^* with ξ^{**}; here we only point out the differences:

- Since now $\gamma=\gamma_2$ and i' is the shared supplier in γ_2, $\gamma\ominus i'=\gamma_1$. Hence by the induction hypothesis

$$\xi^{**}(\gamma\ominus i')=\begin{cases}\text{AD}(i), & \text{if } F<c_{\text{RP}};\\ \text{AR}(i), & \text{if } F\geqslant c_{\text{RP}} \text{ and } a+ur<uwz;\\ \text{PP}, & \text{if } F\geqslant c_{\text{RP}} \text{ and } a+ur\geqslant uwz;\end{cases} \tag{D.96}$$

where $F=\frac{1}{1+u}[a+u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))]+\frac{u}{1+u}(a+u\pi(\gamma\ominus i'\ominus i))$. We redefine the three subcases b(i), b(ii), and b(iii) in the proof by the three cases for $\xi^{**}(\gamma\ominus i')$ in (D.96) (i.e., replace $a+u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))$ in the original condition for each subcase by F).

- In subcases b(ii) and b(iii), owing to (D.86),

$$\nabla(\gamma_1,i')>\nabla(\gamma_1\ominus i,i'). \tag{D.97}$$

$F\geqslant c_{\text{RP}}$ implies $a+u(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i))\geqslant c_{\text{RP}}$.

- In subcase b(iii) ($F\geqslant c_{\text{RP}}$ and $a+ur\geqslant uwz$) conditional on H_{00} the path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma\xrightarrow{\text{AD}(i)}\gamma\ominus i \tag{D.98}$$

while the path of state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^{**} is

$$\gamma\xrightarrow{\text{AD}(i')}\gamma\ominus i'\xrightarrow{\text{PP}}. \tag{D.99}$$

Here by the definition of $\widehat{\xi}$ and the induction hypothesis one of two actions could be taken subsequent to (D.98) ($\pi(\gamma\ominus i\ominus i')=\frac{1}{64}\frac{(\alpha-v_T)^2}{\beta}$ by Proposition C.5):

A. If $a+u\pi(\gamma\ominus i\ominus i')=a+u\frac{1}{64}\frac{(\alpha-v_T)^2}{\beta}\geqslant c_{\text{RP}}$, then $\widehat{\xi}(\gamma\ominus i)=\xi^{**}(\gamma\ominus i\ominus i')=\text{PP}$;

B. If $a+u\pi(\gamma\ominus i\ominus i')=a+u\frac{1}{64}\frac{(\alpha-v_T)^2}{\beta}<c_{\text{RP}}$, then $\widehat{\xi}(\gamma\ominus i)=\xi^{**}(\gamma\ominus i\ominus i')=\text{AD}(i'')$ where i'' is the only supplier in $U_{\gamma\ominus i\ominus i'}$.

In sub-subcase (b)(iii)A the original proof applies. In sub-subcase (b)(iii)B following (D.98) the action is AD(i'') with two possible consequences: that i'' passes the audit leading to $\gamma\ominus i\oplus i''$ and that i'' fails the audit leading to $\gamma\ominus i\ominus i''$. Note that in either case the definition of $\widehat{\xi}$ prescribes PP afterward. Therefore, the expected profit subsequent to $\gamma\ominus i$ in (D.98) is

$$-a+(1-u)\pi(\gamma\ominus i\oplus i'')+u\pi(\gamma\ominus i\ominus i'')-wz=-a+\left[(1-u)\frac{25}{576}+u\frac{1}{36}\right]\frac{(\alpha-v_T)^2}{\beta}-wz \tag{D.100}$$

where the $-wz$ comes from that in event H_{00} we know i' is noncompliant and the equality is by Proposition C.5. On the other hand the expected profit subsequent to $\gamma\ominus i'$ in (D.99) is

$$\pi(\gamma\ominus i')-uwz-wz=\frac{1}{25}\frac{(\alpha-v_T)^2}{\beta}-uwz-wz \tag{D.101}$$

where the $-uwz$ is due to i'' remaining unaudited, the $-wz$ is due to i being noncompliant, and the equality is by Proposition C.5. We take the difference between (D.100) and (D.101) to get

$$-a+uwz-u\frac{1}{64}\frac{(\alpha-v_T)^2}{\beta}+\frac{49}{14,400}\frac{(\alpha-v_T)^2}{\beta}. \tag{D.102}$$

But the premise of the sub-subcase is that $a+u\frac{1}{64}\frac{(\alpha-v_T)^2}{\beta}<c_{\text{RP}}$ where $c_{\text{RP}}=uwz$ here, so (D.102) is nonnegative.

Therefore the expected profit at γ from first taking AD(i) then following $\widehat{\xi}$ is greater than or equal to that from first taking AD(i') then following ξ^{**} conditional on H_{00}. This completes the proof. □

Proof of Corollary 2. The result follows directly from Proposition 2. □

D.3. Supplier Choice When Auditing One Firm

Let $\gamma\in\Gamma$ and $i\in U_\gamma$. We define two thresholds for z:

$$z_p(\gamma,i)=\frac{u\nabla(\gamma,i)+a}{uw[u(|U_\gamma|-|U_{\gamma\ominus i}|-1)+1]}\quad\text{and}\quad z_r(\gamma,i)=\frac{\nabla(\gamma,i)-r}{uw(|U_\gamma|-|U_{\gamma\ominus i}|-1)}. \tag{D.103}$$

PROPOSITION D.3. *At nonterminal state γ, suppose the buyer can at most audit (* AD *or* AR*) one supplier, then* PP*. The optimal decision is*

(*a*) PP *if and only if $z\leqslant\frac{a+ur}{uw}$ and $z\leqslant z_p(\gamma,i')$ for every $i'\in U_\gamma$;*

(*b*) AR(i) *(for any $i\in U_\gamma$) if and only if $z>\frac{a+ur}{uw}$, and $z\leqslant z_r(\gamma,i')$ for every $i'\in U_\gamma$ with an unaudited dependent in γ and $\nabla(\gamma,i')\geqslant r$ for every $i'\in U_\gamma$ without an unaudited dependent in γ.*

(*c*) AD(i) *if and only if $z>z_p(\gamma,i)$, $z>z_r(\gamma,i)$ if i has an unaudited dependent in γ and $\nabla(\gamma,i)<r$ if i does not have an unaudited dependent in γ, and also i satisfies* (D.118).

Proof. Let ξ_{PP} be the policy that maps any state in Γ to the action PP. Then for $i\in U_\gamma$,

$$\begin{aligned}\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i))&=-a+(1-u)V(\xi_{\text{PP}},\gamma\oplus i)+uV(\xi_{\text{PP}},\gamma\ominus i) &\text{(D.104)}\\ &=-a+(1-u)(\pi(\gamma\oplus i)-uwz|U_{\gamma\oplus i}|)+u(\pi(\gamma\ominus i)-uwz|U_{\gamma\ominus i}|) &\text{(D.105)}\\ &=-a+(1-u)[\pi(\gamma)-uwz(|U_\gamma|-1)]+u(\pi(\gamma\ominus i)-uwz|U_{\gamma\ominus i}|) &\text{(D.106)}\end{aligned}$$

$$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i)) = -a - ur + \pi(\gamma\oplus i) - uwz|U_{\gamma\oplus i}| \tag{D.107}$$

$$= -a - ur + \pi(\gamma) - uwz(|U_\gamma|-1) \tag{D.108}$$

and

$$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP}) = \pi(\gamma) - uwz|U_\gamma|. \tag{D.109}$$

Note that $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i))$ is independent of i.

Therefore $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i'))$ for any $i' \in U_\gamma$ iff

$$-a + (1-u)[\pi(\gamma) - uwz(|U_\gamma|-1)] + u(\pi(\gamma\ominus i) - uwz|U_{\gamma\ominus i}|) > -a - ur + \pi(\gamma) - uwz(|U_\gamma|-1) \tag{D.110}$$

which is equivalent to

$$uwz(|U_\gamma| - |U_{\gamma\ominus i}| - 1) > \pi(\gamma) - \pi(\gamma\ominus i) - r. \tag{D.111}$$

If i has no unaudited dependent, i.e., $D_g(i)\cap U_\gamma = \emptyset$ where g is the supply network in state γ, then $|U_\gamma|-1 = |U_{\gamma\ominus i}|$; (D.111) is equivalent to $\pi(\gamma)-\pi(\gamma\ominus i) < r$. Otherwise if i has at least one unaudited dependent, then $|U_\gamma|-1 > |U_{\gamma\ominus i}|$; (D.111) is equivalent to

$$z > \frac{\pi(\gamma)-\pi(\gamma\ominus i) - r}{uw(|U_\gamma| - |U_{\gamma\ominus i}| - 1)}. \tag{D.112}$$

$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$ iff

$$-a + (1-u)[\pi(\gamma) - uwz(|U_\gamma - 1|)] + u(\pi(\gamma\ominus i) - uwz|U_{\gamma\ominus i}|) > \pi(\gamma) - uwz|U_\gamma| \tag{D.113}$$

which is equivalent to

$$z > \frac{u(\pi(\gamma)-\pi(\gamma\ominus i)) + a}{uw[u(|U_\gamma| - |U_{\gamma\ominus i}|) + 1 - u]}. \tag{D.114}$$

$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i'))$ for $i' \in U_\gamma$ iff

$$\begin{aligned}&-a + (1-u)[\pi(\gamma) - uwz(|U_\gamma|-1)] + u(\pi(\gamma\ominus i) - uwz|U_{\gamma\ominus i}|)\\ &\qquad \geqslant -a + (1-u)[\pi(\gamma) - uwz(|U_\gamma|-1)] + u(\pi(\gamma\ominus i') - uwz|U_{\gamma\ominus i'}|)\end{aligned} \tag{D.115}$$

which is equivalent to

$$\pi(\gamma\ominus i) - uwz|U_{\gamma\ominus i}| \geqslant \pi(\gamma\ominus i') - uwz|U_{\gamma\ominus i'}|. \tag{D.116}$$

$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$ iff

$$-a - ur + \pi(\gamma) - uwz(|U_\gamma|-1) > \pi(\gamma) - uwz|U_\gamma| \tag{D.117}$$

which is equivalent to $z > \frac{a+ur}{uw}$.

The optimal decision is PP iff $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP}) \geqslant \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i))$ and $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP}) \geqslant \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i))$ for any $i \in U_\gamma$. This gives part (a). The optimal decision is AD(i) iff $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i'))$, $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$, and $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i'))$ for any $i' \in U_\gamma$. This gives part (c). The optimal decision is AR(i) iff $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i'))$ for any $i' \in U_\gamma$ and $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$. This gives part (b). □

PROPOSITION D.4. *At nonterminal state γ, suppose the buyer can audit (*AD *or* AR*) at most one supplier, before proceeding to production (*PP*). There exist two (possibly coinciding) thresholds $\underline{z} \leqslant \overline{z}$ for penalty z such that*

(*a*) *If $z \leqslant \underline{z}$ the optimal decision is* PP*;*

(*b*) *If $\underline{z} < z \leqslant \overline{z}$ the optimal decision is* AR(i) *for any $i \in U_\gamma$;*

(*c*) *If $z > \overline{z}$ the optimal decision is* AD(i) *where i solves*

$$\max_{i \in U_\gamma} \{uwz(|U_\gamma| - |U_{\gamma \ominus i}|) - \nabla(\gamma, i)\} \tag{D.118}$$

Proof. By Proposition D.3 the optimal decision depends on the ordering of three thresholds for z:

$$\frac{a+ur}{uw}, \quad \underline{z}_r(\gamma) = \min\{z_r(\gamma, i) : i \in U_\gamma \text{ with unaudited dependent}\}, \quad \underline{z}_p(\gamma) = \min_{i \in U_\gamma} z_p(\gamma, i). \tag{D.119}$$

In the following we enumerate all but one possible orderings of the three thresholds to verify that they are consistent with the property we describe in Proposition D.4. We then show the remaining one ordering can never arise. In the following the supplier i in AR(i) can be any $i \in U_\gamma$ and the supplier i in AD(i) is given by (D.118).[12] We consider two mutually exclusive and collectively exhaustive cases as follows.

(*a*) First we look at the case that either $\nabla(\gamma, i') < r$ for some $i' \in U_\gamma$ without an unaudited dependent, or $\frac{a+ur}{uw} \geqslant \underline{z}_r(\gamma)$. Then by Proposition D.3 AR(i) is never optimal. Therefore the optimal decision is either PP or AD(i). By Proposition D.3 the optimal decision is PP if and only if $z \leqslant \left(\frac{a+ur}{uw}\right) \wedge \underline{z}_p(\gamma)$, which implies the optimal decision is AD(i) if and only if $z > \left(\frac{a+ur}{uw}\right) \wedge \underline{z}_p(\gamma)$. Setting $\underline{z} = \overline{z} = \left(\frac{a+ur}{uw}\right) \wedge \underline{z}_p(\gamma)$ establishes the property Proposition D.4 describes.

(*b*) Second we look at the case that $\nabla(\gamma, i') \geqslant r$ for every $i' \in U_\gamma$ without an unaudited dependent, and $\frac{a+ur}{uw} < \underline{z}_r(\gamma)$. By Proposition D.3 the optimal decision is AR(i) if and only if $\frac{a+ur}{uw} < z \leqslant \underline{z}_r(\gamma)$. Suppose $\frac{a+ur}{uw} \leqslant \underline{z}_p(\gamma)$ then the optimal decision is PP if and only if $z \leqslant \frac{a+ur}{uw}$. So setting $\underline{z} = \frac{a+ur}{uw}$ and $\overline{z} = \underline{z}_r(\gamma)$ will establish the property Proposition D.4 describes. We only need to show that indeed $\frac{a+ur}{uw} \leqslant \underline{z}_p(\gamma)$ under case (*b*).

By way of contradiction suppose $\frac{a+ur}{uw} > \underline{z}_p(\gamma)$. It implies that there exists $i'' \in U_\gamma$ such that

$$z_p(\gamma, i'') = \frac{u\nabla(\gamma, i'') + a}{uw[u(|U_\gamma| - |U_{\gamma \ominus i''}| - 1) + 1]} < \frac{a+ur}{uw}. \tag{D.120}$$

- If i'' does not have an unaudited dependent in γ, then $|U_{\gamma \ominus i''}| = |U_\gamma| - 1$; then (D.120) implies $\nabla(\gamma, i'') < r$, contradicting that $\nabla(\gamma, i') \geqslant r$ for every $i' \in U_\gamma$ without an unaudited dependent (in the premise of case (*b*)).
- If i'' has an unaudited dependent, then (D.120) implies

$$\nabla(\gamma, i'') < (a+ur)(|U_\gamma| - |U_{\gamma \ominus i''}| - 1) + r. \tag{D.121}$$

On the other hand, given that i'' has an unaudited dependent in γ, $\frac{a+ur}{uw} \leqslant \underline{z}_r(\gamma)$ implies

$$\frac{a+ur}{uw} \leqslant z_r(\gamma, i'') = \frac{\nabla(\gamma, i'') - r}{uw(|U_\gamma| - |U_{\gamma \ominus i''}| - 1)} \tag{D.122}$$

which is equivalent to

$$\nabla(\gamma, i'') - r \geqslant (a+ur)(|U_\gamma| - |U_{\gamma \ominus i''}| - 1) \tag{D.123}$$

contradicting (D.121). Therefore under case (*b*), $\frac{a+ur}{uw} \leqslant \underline{z}_p(\gamma)$. □

[12] The identity of supplier i may change as z changes.

We consider a state $\gamma_+ = (g, U)$ in which there is at least one supplier in each position in tier 2 (majority-exclusive, minority-exclusive, shared; i.e., $t_{\text{A}}, t_{\text{B}}, t_{\text{AB}} \geqslant 1$), all suppliers (including those in tier 1) are unaudited, and the majority tier-1 firm A has strictly more suppliers than the minority tier-1 firm B (i.e., $t_{\text{A}} > t_{\text{B}}$). This structure allows us to compare all possible auditing choices.

We define the following thresholds used in Proposition D.5:

$$z_{\text{A}|\text{B}} = \frac{1}{16}\frac{(\alpha - v_T)^2}{uw\beta}\frac{1}{t_{\text{A}} - t_{\text{B}}}\left[\left(\frac{t_{\text{A}} + t_{\text{AB}}}{t_{\text{A}} + t_{\text{AB}} + 1}\right)^2 - \left(\frac{t_{\text{B}} + t_{\text{AB}}}{t_{\text{B}} + t_{\text{AB}} + 1}\right)^2\right] \tag{D.124}$$

$$z_{\text{B}|1} = \frac{(\alpha - v_T)^2}{uw\beta}\frac{1}{t_{\text{B}}}\left[\frac{1}{9}Y - \frac{1}{16}\left(\frac{t_{\text{A}} + t_{\text{AB}}}{t_{\text{A}} + t_{\text{AB}} + 1}\right)^2\right] \tag{D.125}$$

$$z_{\text{A}|1} = \frac{(\alpha - v_T)^2}{uw\beta}\frac{1}{t_{\text{A}}}\left[\frac{1}{9}Y - \frac{1}{16}\left(\frac{t_{\text{B}} + t_{\text{AB}}}{t_{\text{B}} + t_{\text{AB}} + 1}\right)^2\right] \tag{D.126}$$

where

$$Y = \begin{cases} \left[\dfrac{3(t_{\text{A}} - 1) + 3t_{\text{B}} + 4t_{\text{AB}} + 4(t_{\text{A}} - 1)t_{\text{AB}} + 4t_{\text{B}}t_{\text{AB}} + 3(t_{\text{A}} - 1)t_{\text{B}} + 4t_{\text{AB}}^2}{L(t_{\text{A}} - 1, t_{\text{B}}, t_{\text{AB}})}\right]^2, & \text{if } t_{\text{A}} \leqslant 2t_{\text{B}} + 2t_{\text{AB}} + 2 \\ \left[\dfrac{3(t_{\text{A}} - 1) + 3(t_{\text{B}} + t_{\text{AB}}) + 3(t_{\text{A}} - 1)(t_{\text{B}} + t_{\text{AB}})}{L(t_{\text{A}} - 1, t_{\text{B}} + t_{\text{AB}}, 0)}\right]^2, & \text{if } t_{\text{A}} > 2t_{\text{B}} + 2t_{\text{AB}} + 2 \end{cases}. \tag{D.127}$$

Proposition D.5. *At state γ_+ suppose the buyer can audit at most one supplier before proceeding to production (PP). Let $\underline{z}_d = (z_{\text{A}|1} \wedge z_{\text{B}|1}) \vee \bar{z}$ and $\bar{z}_d = z_{\text{A}|1} \vee z_{\text{A}|\text{B}} \vee \bar{z}$ where $\bar{z}$ is as in Proposition D.4. The optimal decision is AD($\mathbb{e}_{\text{A}}$) (i.e., auditing and dropping (if noncompliant) an exclusive supplier to firm A) if and only if $\bar{z} < z \leqslant \underline{z}_d$, AD(B) if and only if $\underline{z}_d < z \leqslant \bar{z}_d$, and AD(A) if and only if $z > \bar{z}_d$.*

Proof. By the proof of Proposition D.3, let $i, i' \in U_\gamma$, $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i'))$ iff

$$\pi(\gamma \ominus i) - uwz|U_{\gamma\ominus i}| \geqslant \pi(\gamma \ominus i') - uwz|U_{\gamma\ominus i'}|. \tag{D.128}$$

Since no tier-2 supplier has a dependent in γ, (D.128) implies the buyer should prefer among tier-2 suppliers to AD a supplier i with the highest $\pi(\gamma \ominus i)$. By Proposition 2 this supplier is a majority-exclusive supplier. Hence we only need to compare the majority-exclusive supplier 1, firm A, and firm B.

By the structure of supply network g,

$$|U_{\gamma\ominus\text{A}}| = |U_\gamma| - t_{\text{A}} - 1 \tag{D.129}$$

$$|U_{\gamma\ominus\text{B}}| = |U_\gamma| - t_{\text{B}} - 1 \tag{D.130}$$

$$|U_{\gamma\ominus 1}| = |U_\gamma| - 1. \tag{D.131}$$

By Proposition C.5,

$$\pi(\gamma \ominus \text{A}) = \frac{1}{16}\frac{(\alpha - v_T)^2}{\beta}\left(\frac{t_{\text{B}} + t_{\text{AB}}}{t_{\text{B}} + t_{\text{AB}} + 1}\right)^2 \tag{D.132}$$

$$\pi(\gamma \ominus \text{B}) = \frac{1}{16}\frac{(\alpha - v_T)^2}{\beta}\left(\frac{t_{\text{A}} + t_{\text{AB}}}{t_{\text{A}} + t_{\text{AB}} + 1}\right)^2 \tag{D.133}$$

$$\pi(\gamma \ominus 1) = \frac{1}{9}\frac{(\alpha - v_T)^2}{\beta}Y. \tag{D.134}$$

One may verify that the buyer prefers AD(A) to AD(B) if and only if $z \geqslant z_{\text{A}|\text{B}}$, the buyer prefers AD(B) to AD(1) if and only if $z \geqslant z_{\text{B}|1}$, and the buyer prefers AD(A) to AD(1) if and only if $z \geqslant z_{\text{A}|1}$, by plugging (D.129)–(D.134) into (D.128). By Proposition D.4 when $z > \hat{z}$ the optimal decision is to AD some supplier. Proposition D.5 now follows. □

Appendix E: Corresponding Results When Adopting ABM Model

We show that all our results of the auditing phase hold when we replace our production model with the corresponding case of the model in Adida et al. (2016) (ABM hereafter) except that our model additionally captures the difference in the value that a majority-exclusive and a minority-exclusive supplier provide. Specifically we redefine the production profit $\pi(\gamma) = \pi^*_{\text{C}}$ to be the retailer's profit (π_r in the notation of ABM) in the model in Section 5 of ABM for the case where there is one buyer (retailer) and two tier-1 firms (intermediaries).

PROPOSITION E.1. *The buyer's production profit is*

$$\pi^*_{\text{C}} = \left(\frac{d_1 - s_1}{2}\right)^2 \frac{|S(2)|\bar{I}}{d_2|S(2)|\bar{I} + s_2(\bar{I}+1)} \tag{E.1}$$

where d_1, d_2, s_1, s_2 are demand and cost parameters defined in ABM and

$$\bar{I} = \left\{1 - \frac{1}{|S(2)|}\left[\frac{1}{2}(t_{\text{A}} + t_{\text{B}}) + \frac{2}{3}t_{\text{AB}}\right]\right\}^{-1} - 1 \tag{E.2}$$

if $I = 1$ or 2 and $\bar{I} = 0$ if $I = 0$.

Proof. This is a direct consequence of Theorem 3 and Proposition 1 in ABM. In particular, when there are two tier-1 firms (E.2) is derived in the same way as is equation (A17) in ABM. When there is only one tier-1 firm, by Proposition 1 in ABM and since each tier-2 supplier now serves only one customer, i.e., $I_j = 1$ for all tier-2 suppliers j,

$$\frac{\bar{I}}{\bar{I}+1} = \frac{1}{|S(2)|}|S(2)|\frac{1}{1+1} \tag{E.3}$$

Therefore $\bar{I} = 1$, which coincides with (E.2) by setting $t_{\text{B}} = t_{\text{AB}} = 0$. □

REMARK E.1. By (E.1) and (E.2) any exclusive supplier, whether it supplies firms A or B, carries the same value to the buyer in π^*_{C}. That is π^*_{C} stays the same however t_{A} and t_{B} vary as long as $t_{\text{A}} + t_{\text{B}}$ remains the same.

COROLLARY E.1. *Given supply network $g \in G$,*

(a) *Adding any supplier to g increases the buyer's production profit π^*_{C};*

(b) *Adding a shared supplier to g induces a strictly greater increase in π^*_{C} than adding an exclusive supplier.*

Proof. (Part (a)) By (E.1) and algebra

$$\begin{aligned}\Delta_1\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle &= \Delta_2\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle \\ &= \frac{9s_2(d_1 - s_1)^2}{2[3d_2(t_{\text{A}} + t_{\text{B}}) + 4d_2t_{\text{AB}} + 6s_2][3d_2(t_{\text{A}} + t_{\text{B}}) + 4d_2t_{\text{AB}} + 3d_2 + 6s_2]} > 0\end{aligned} \tag{E.4}$$

by Assumption 1 in ABM, and

$$\Delta_3\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle = \frac{12s_2(d_1 - s_1)^2}{2[3d_2(t_{\text{A}} + t_{\text{B}}) + 4d_2t_{\text{AB}} + 6s_2][3d_2(t_{\text{A}} + t_{\text{B}}) + 4d_2t_{\text{AB}} + 4d_2 + 6s_2]} > 0 \tag{E.5}$$

by Assumption 1 in ABM.

(Part (b)) By Remark E.1 we only need to show

$$\Delta_3\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle - \Delta_1\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle > 0 \tag{E.6}$$

which is equivalent to

$$\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}+1\rangle > \pi^*_{\text{C}}\langle t_{\text{A}}+1, t_{\text{B}}, t_{\text{AB}}\rangle. \tag{E.7}$$

Now we can equivalently write (E.1) as

$$\pi^*_{\text{C}} = \left(\frac{d_1 - s_1}{2}\right)^2 \frac{|S(2)|\bar{I}}{(d_2|S(2)| + s_2)\bar{I} + s_2} \tag{E.8}$$

which, with everything else kept constant, is strictly increasing in $\bar{I}$. Since

$$\omega \equiv \frac{1}{|S(2)|}\left[\frac{1}{2}(t_{\text{A}} + t_{\text{B}}) + \frac{2}{3}t_{\text{AB}}\right] \in (0,1) \tag{E.9}$$

by (E.2) $\bar{I}$ is strictly increasing in ω. Lastly, ω clearly increases strictly more by adding 1 to t_{AB} than by adding 1 to t_{A}, which is to say that (E.7) holds. □

The statements and proofs of our Theorem 3 and Corollary 1 remain the same as in the base model.

Proposition E.2 **(decreasing differences of production profit)**. *For any $\gamma \in \Gamma$ and any $i, i' \in U_\gamma$, $i' \notin D_\gamma(i)$,*

$$\nabla(\gamma, i') \leqslant \nabla(\gamma \ominus i, i'). \tag{E.10}$$

Proof. Given Remark E.1 we only need to show each of the following four differences is nonpositive:

$$\Delta_1\pi^*_{\text{C}}\langle t_{\text{A}}+1, t_{\text{B}}, t_{\text{AB}}\rangle - \Delta_1\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle \tag{E.11}$$

$$\Delta_1\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}+1\rangle - \Delta_1\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle \tag{E.12}$$

$$\Delta_3\pi^*_{\text{C}}\langle t_{\text{A}}+1, t_{\text{B}}, t_{\text{AB}}\rangle - \Delta_3\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle \tag{E.13}$$

$$\Delta_3\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}+1\rangle - \Delta_3\pi^*_{\text{C}}\langle t_{\text{A}}, t_{\text{B}}, t_{\text{AB}}\rangle. \tag{E.14}$$

Using (E.1) by algebra we find that they are respectively equal to

$$-\frac{1}{3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 6s_2} \frac{27d_2s_2(d_1-s_1)^2}{[3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 3d_2 + 6s_2][3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 6d_2 + 6s_2]} \tag{E.15}$$

$$-\frac{1}{[3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 6s_2][3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 3d_2 + 6s_2]} \frac{18d_2s_2(d_1-s_1)^2[6d_2(t_{\text{A}}+t_{\text{B}}) + 8d_2t_{\text{AB}} + 7d_2 + 12s_2]}{[3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 4d_2 + 6s_2][3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 7d_2 + 6s_2]} \tag{E.16}$$

$$-\frac{1}{[3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 6s_2][3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 3d_2 + 6s_2]} \frac{18d_2s_2(d_1-s_1)^2[6d_2(t_{\text{A}}+t_{\text{B}}) + 8d_2t_{\text{AB}} + 7d_2 + 12s_2]}{[3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 4d_2 + 6s_2][3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 7d_2 + 6s_2]} \tag{E.17}$$

$$-\frac{1}{3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 6s_2} \frac{48d_2s_2(d_1-s_1)^2}{[3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 4d_2 + 6s_2][3d_2(t_{\text{A}}+t_{\text{B}}) + 4d_2t_{\text{AB}} + 8d_2 + 6s_2]} \tag{E.18}$$

which are obviously all nonpositive (in fact strictly negative by Assumption 1 in ABM). □

THEOREM E.1. *The following policy ξ^{**} is optimal at any state γ in which every tier-1 firm is vetted: for any nonterminal state $\gamma \in \Gamma$, let i be an LVUS in γ, then*

$$\xi^{**}(\gamma) = \begin{cases} \text{AD}(i), & \text{if } u\nabla(\gamma, i) + a < c_{\text{RP}} \\ \text{RP}, & \text{if } u\nabla(\gamma, i) + a \geqslant c_{\text{RP}} \end{cases}. \tag{E.19}$$

Proof. By Proposition E.2 Assumption D.1 holds. Given Remark E.1 Assumption D.2 clearly holds. Then Theorem E.1 is a direct consequence of Theorem D.1. □

The proofs of Propositions D.3 and D.4 are the same as in our model. I will next state and prove the result that corresponds to Proposition D.5.

We define the following thresholds used in Proposition E.3:

$$z_{\text{A}|\text{B}} = \frac{1}{uw(t_{\text{A}} - t_{\text{B}})} \left(\frac{d_1 - s_1}{2}\right)^2 \left[\frac{t_{\text{A}} + t_{\text{AB}}}{d_2(t_{\text{A}} + t_{\text{AB}}) + 2s_2} - \frac{t_{\text{B}} + t_{\text{AB}}}{d_2(t_{\text{B}} + t_{\text{AB}}) + 2s_2}\right] \tag{E.20}$$

$$z_{\text{B}|1} = \frac{1}{uwt_{\text{B}}} \left(\frac{d_1 - s_1}{2}\right)^2 \left\{\frac{\frac{1}{2}(t_{\text{A}} + t_{\text{B}} - 1) + \frac{2}{3}t_{\text{AB}}}{d_2\left[\frac{1}{2}(t_{\text{A}} + t_{\text{B}} - 1) + \frac{2}{3}t_{\text{AB}}\right] + s_2} - \frac{t_{\text{A}} + t_{\text{AB}}}{2[d_2(t_{\text{A}} + t_{\text{AB}}) + 2s_2]}\right\} \tag{E.21}$$

$$z_{\text{A}|1} = \frac{1}{uwt_{\text{A}}} \left(\frac{d_1 - s_1}{2}\right)^2 \left\{\frac{\frac{1}{2}(t_{\text{A}} + t_{\text{B}} - 1) + \frac{2}{3}t_{\text{AB}}}{d_2\left[\frac{1}{2}(t_{\text{A}} + t_{\text{B}} - 1) + \frac{2}{3}t_{\text{AB}}\right] + s_2} - \frac{t_{\text{B}} + t_{\text{AB}}}{2[d_2(t_{\text{B}} + t_{\text{AB}}) + 2s_2]}\right\}. \tag{E.22}$$

PROPOSITION E.3. *At state γ_+ suppose the buyer can audit at most one supplier before proceeding to production (PP). Let $\underline{z}_d = (z_{\text{A}|1} \wedge z_{\text{B}|1}) \vee \bar{z}$ and $\bar{z}_d = z_{\text{A}|1} \vee z_{\text{A}|\text{B}} \vee \bar{z}$ where $\bar{z}$ is as in Proposition 7. The optimal decisions are AD($\oplus_{\text{A}}$) and AD($\oplus_{\text{B}}$) (i.e., auditing and dropping (if noncompliant) an exclusive supplier) if and only if $\bar{z} < z \leqslant \underline{z}_d$, AD(B) if and only if $\underline{z}_d < z \leqslant \bar{z}_d$, and AD(A) if and only if $z > \bar{z}_d$.*

Proof of Proposition E.3. By the proof of Proposition D.3 let $i, i' \in U_\gamma$, $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i'))$ iff

$$\pi(\gamma \ominus i) - uwz|U_{\gamma \ominus i}| \geqslant \pi(\gamma \ominus i') - uwz|U_{\gamma \ominus i'}|. \tag{E.23}$$

Since no tier-2 supplier has a dependent in γ, (E.23) implies the buyer should prefer among tier-2 suppliers to AD a supplier i with the highest $\pi(\gamma \ominus i)$. By Remark E.1 and Corollary E.1 this supplier is any exclusive supplier. Hence we only need to compare the majority-exclusive supplier 1, firm A, and firm B.

By the structure of supply network g,

$$|U_{\gamma \ominus \text{A}}| = |U_\gamma| - t_{\text{A}} - 1 \tag{E.24}$$

$$|U_{\gamma \ominus \text{B}}| = |U_\gamma| - t_{\text{B}} - 1 \tag{E.25}$$

$$|U_{\gamma \ominus 1}| = |U_\gamma| - 1. \tag{E.26}$$

By Proposition E.1,

$$\pi(\gamma \ominus \text{A}) = \left(\frac{d_1 - s_1}{2}\right)^2 \frac{t_{\text{B}} + t_{\text{AB}}}{d_2(t_{\text{B}} + t_{\text{AB}}) + 2s_2} \tag{E.27}$$

$$\pi(\gamma \ominus \text{B}) = \left(\frac{d_1 - s_1}{2}\right)^2 \frac{t_{\text{A}} + t_{\text{AB}}}{d_2(t_{\text{A}} + t_{\text{AB}}) + 2s_2} \tag{E.28}$$

$$\pi(\gamma \ominus 1) = \left(\frac{d_1 - s_1}{2}\right)^2 \frac{\frac{1}{2}(t_{\text{A}} + t_{\text{B}} - 1) + \frac{2}{3}t_{\text{AB}}}{d_2\left[\frac{1}{2}(t_{\text{A}} + t_{\text{B}} - 1) + \frac{2}{3}t_{\text{AB}}\right] + s_2}. \tag{E.29}$$

One may verify that the buyer prefers AD(A) to AD(B) if and only if $z \geqslant z_{\text{A}|\text{B}}$, the buyer prefers AD(B) to AD(1) if and only if $z \geqslant z_{\text{B}|1}$, and the buyer prefers AD(A) to AD(1) if and only if $z \geqslant z_{\text{A}|1}$, by plugging (E.24)–(E.29) into (E.23). By Proposition D.4 when $z > \widehat{z}$ the optimal decision is to AD some supplier. Proposition E.3 now follows. □

Appendix F: Heterogeneous Penalty Across Tiers: Specification and Results

Assume the buyer incurs a cost of $z_1 \geqslant 0$ upon the exposure of a violation at each noncompliant supplier in tier 1, and a cost of $z_2 \geqslant 0$ at each in tier 2. Then by the independence of noncompliance and of the exposure of violations across suppliers, $\zeta(\gamma) = |U_\gamma \cap S(1)|uwz_1 + |U_\gamma \cap S(2)|uwz_2$. Denote $U_\gamma^{(k)} = U_\gamma \cap S(k)$, $k = 1, 2$.

F.1. Two Subphases of Auditing

Proposition D.1 and its proof remain the same. Proposition D.2 is replaced by the following:

Proposition F.1. *Limit the buyer's actions to auditing and rectifying (AR) unaudited suppliers and proceeding to production (PP). An optimal auditing policy is the following:*

(*a*) *If $a + ur \leqslant uwz_1$ and $a + ur \leqslant uwz_2$, audit and rectify (AR) all remaining unaudited suppliers in any sequence, then proceed to production (PP);*

(*b*) *If $a + ur \leqslant uwz_1$ and $a + ur > uwz_2$, audit and rectify (AR) all remaining unaudited suppliers in tier 1 in any sequence, then proceed to production (PP);*

(*c*) *If $a + ur > uwz_1$ and $a + ur \leqslant uwz_2$, audit and rectify (AR) all remaining unaudited suppliers in tier 2 in any sequence, then proceed to production (PP);*

(*d*) *If $a + ur > uwz_1$ and $a + ur > uwz_2$, proceed to production directly (PP).*

Furthermore, given $\gamma \in \Gamma$,

$$V^*(\gamma) = \pi(\gamma) - [(uwz_1) \wedge (a + ur)]|U_\gamma \cap S(1)| - [(uwz_2) \wedge (a + ur)]|U_\gamma \cap S(2)|. \tag{F.1}$$

Proof. Given $\gamma \in \Gamma$ and $i \in U_\gamma$, by (2) and the definition of ζ,

$$\widetilde{V}^*(\gamma, \text{PP}) = \pi(\gamma) - |U_\gamma \cap S(1)|uwz_1 - |U_\gamma \cap S(2)|uwz_2. \tag{F.2}$$

Analogous to the proof of Proposition D.2 we proceed by mathematical induction on the number of unaudited supplier in the state, $|U_\gamma|$. If $|U_\gamma| = 1$, let $i \in U_\gamma$, then $\gamma \oplus i$ is a terminal state. By (5),

$$\widetilde{V}^*(\gamma, \text{AR}(i)) = -a - ur + V^*(\gamma \oplus i) = -a - ur + \pi(\gamma \oplus i) = -a - ur + \pi(\gamma) \tag{F.3}$$

since states $\gamma \oplus i$ and γ have the same underlying supply network, which determines the production profit. Denote by k the tier that i belongs to, i.e., $i \in S(k)$. The buyer chooses $\text{AR}(i)$ over PP if and only if $\widetilde{V}^*(\gamma, \text{AR}(i)) \geqslant \widetilde{V}^*(\gamma, \text{PP})$, or $-a - ur + \pi(\gamma) \geqslant \pi(\gamma) - uwz_k$, or $a + ur \leqslant uwz_k$, confirming the policy in Proposition F.1. As a consequence $V^*(\gamma) = \pi(\gamma) - (uwz_k) \wedge (a + ur)$, confirming (F.1).

By mathematical induction, suppose for any state γ such that $|U_\gamma| = m$, $V^*(\gamma)$ is given by (F.1). Now given any state γ such that $|U_\gamma| = m + 1$, pick arbitrary $i \in U_\gamma$, then $|U_{\gamma \oplus i}| = m$. Suppose i is in tier $k \in \{1, 2\}$. Let $|U_\gamma \cap S(1)| = m_1$ and $|U_\gamma \cap S(2)| = m_2$. By (5) and (F.1),

$$\widetilde{V}^*(\gamma, \text{AR}(i)) = -a - ur + V^*(\gamma \oplus i) \tag{F.4}$$

$$= -a - ur + \pi(\gamma) - m_1[(uwz_1) \wedge (a + ur)]$$

$$- m_2[(uwz_2) \wedge (a + ur)] + (uwz_k) \wedge (a + ur). \tag{F.5}$$

Note that by (F.5) suppliers i within the same tier give the same $\widetilde{V}^*(\gamma, \text{AR}(i))$. If there are unaudited suppliers in both tier 1 and tier 2, let $i_1 \in U_\gamma \cap S(1)$ and $i_2 \in U_\gamma \cap S(2)$. $\text{AR}(i_1)$ is preferred to $\text{AR}(i_2)$ if and only if $\widetilde{V}^*(\gamma, \text{AR}(i_1)) \geqslant \widetilde{V}^*(\gamma, \text{AR}(i_2))$, or by (F.5),

$$(uwz_1) \wedge (a+ur) \geqslant (uwz_2) \wedge (a+ur). \tag{F.6}$$

If $a+ur \leqslant uwz_1$ and $a+ur \leqslant uwz_2$ then the two sides of (F.6) are equal. In this case the buyer is indifferent between $\text{AR}(i_1)$ and $\text{AR}(i_2)$ in state γ. Otherwise if $a+ur > uwz_1$ or $a+ur > uwz_2$ let k be the tier with higher penalty and k' be the tier with lower penalty, i.e., $z_k \geqslant z_{k'}$, then the buyer prefers $\text{AR}(i_k)$ to $\text{AR}(i_{k'})$.

For later use, we observe that by (F.5) and (2) the inequality $\widetilde{V}^*(\gamma, \text{AR}(i)) \geqslant \widetilde{V}^*(\gamma, \text{PP})$, where $i \in U_\gamma$, expands to

$$\begin{aligned} &-a-ur+\pi(\gamma)-m_1[(uwz_1)\wedge(a+ur)]-m_2[(uwz_2)\wedge(a+ur)]+(uwz_k)\wedge(a+ur) \\ &\qquad \geqslant \pi(\gamma)-m_1(uwz_1)-m_2(uwz_2) \end{aligned} \tag{F.7}$$

which is equivalent to

$$m_1[(uwz_1)-(a+ur)]^+ + m_2[(uwz_2)-(a+ur)]^+ \geqslant [(a+ur)-(uwz_k)]^+. \tag{F.8}$$

Let k and k' represent the two tiers, i.e., $\{k,k'\} = \{1,2\}$. If only tier k has unaudited suppliers (i.e., $U_\gamma \cap S(k) \neq \emptyset$, $U_\gamma \cap S(k') = \emptyset$), then $m_{k'} = 0$. Let $i \in U_\gamma \cap S(k)$. (F.8) reduces to $m_k[(uwz_k)-(a+ur)]^+ \geqslant [(a+ur)-(uwz_k)]^+$, which holds if and only if $a+ur \leqslant uwz_k$: this confirms the policy in Proposition F.1. If both tiers have unaudited suppliers (i.e., $U_\gamma \cap S(k) \neq \emptyset$, $U_\gamma \cap S(k') \neq \emptyset$), pick $i_k \in U_\gamma \cap S(k)$ and $i_{k'} \in U_\gamma \cap S(k')$.

(*a*) If $a+ur \leqslant uwz_k$ and $a+ur \leqslant uwz_{k'}$, we already know that $\widetilde{V}^*(\gamma, \text{AR}(i_k)) = \widetilde{V}^*(\gamma, \text{AR}(i_{k'}))$, and only need to show $\widetilde{V}^*(\gamma, \text{AR}(i_k)) \geqslant \widetilde{V}^*(\gamma, \text{PP})$ to confirm the optimality of the policy in Proposition F.1. This is true since now the right-hand side of (F.8) is zero while the left-hand side is always nonnegative.

(*b*) If $a+ur \leqslant uwz_k$ and $a+ur > uwz_{k'}$, then $z_k > z_{k'}$ and from (F.6) we know that $\widetilde{V}^*(\gamma, \text{AR}(i_k)) \geqslant \widetilde{V}^*(\gamma, \text{AR}(i_{k'}))$. We only need to show that $\widetilde{V}^*(\gamma, \text{AR}(i_k)) \geqslant \widetilde{V}^*(\gamma, \text{PP})$. This is again true since the right-hand side of (F.8) reduces to zero.

(*c*) If $a+ur > uwz_k$ and $a+ur > uwz_{k'}$, without loss of generality, let $z_k \geqslant z_{k'}$. From (F.6) we know that $\widetilde{V}^*(\gamma, \text{AR}(i_k)) \geqslant \widetilde{V}^*(\gamma, \text{AR}(i_{k'}))$. We only need to show that $\widetilde{V}^*(\gamma, \text{AR}(i_k)) < \widetilde{V}^*(\gamma, \text{PP})$. Note that now the right-hand side of (F.8) is strictly positive and the left-hand side is zero: (F.8) cannot be true so indeed $\widetilde{V}^*(\gamma, \text{AR}(i_k)) < \widetilde{V}^*(\gamma, \text{PP})$.

By now we have proved the optimality of the policy in Proposition F.1 in state γ. Finally, to complete the induction step, let $i \in U_\gamma$, $i_k \in U_\gamma \cap S(k)$ and $i_{k'} \in U_\gamma \cap S(k')$ when they each exist:

$$V^*(\gamma) = \begin{cases} \widetilde{V}^*(\gamma, \text{AR}(i)), & \text{if } a+ur \leqslant uw(z_k \wedge z_{k'}); \\ \widetilde{V}^*(\gamma, \text{AR}(i_k)), & \begin{array}{l}\text{if } a+ur \leqslant uwz_k,\ a+ur > uwz_{k'}, \\ \text{and } U_\gamma \cap S(k) \neq \emptyset, \text{ where } k \in \{1,2\};\end{array} \\ \widetilde{V}^*(\gamma, \text{PP}), & \text{otherwise.} \end{cases} \tag{F.9}$$

$$= \begin{cases} -a-ur+\pi(\gamma)-m_1[(uwz_1)\wedge(a+ur)] & \\ -m_2[(uwz_2)\wedge(a+ur)]+(uwz_k)\wedge(a+ur), & \text{if } a+ur\leqslant uw(z_k\wedge z_{k'}); \\ -a-ur+\pi(\gamma)-m_1[(uwz_1)\wedge(a+ur)] & \text{if } a+ur\leqslant uwz_k,\ a+ur>uwz_{k'}, \\ -m_2[(uwz_2)\wedge(a+ur)]+(uwz_k)\wedge(a+ur), & \text{and } U_\gamma\cap S(k)\neq\emptyset,\ \text{where } k\in\{1,2\}; \\ \pi(\gamma)-|U_\gamma\cap S(1)|uwz_1-|U_\gamma\cap S(2)|uwz_2, & \text{otherwise.} \end{cases} \tag{F.10}$$

$$= \begin{cases} \pi(\gamma)-m_1[(uwz_1)\wedge(a+ur)] & \\ -m_2[(uwz_2)\wedge(a+ur)]+(a+ur)-a-ur, & \text{if } a+ur\leqslant uw(z_k\wedge z_{k'}); \\ \pi(\gamma)-m_1[(uwz_1)\wedge(a+ur)] & \text{if } a+ur\leqslant uwz_k,\ a+ur>uwz_{k'}, \\ -m_2[(uwz_2)\wedge(a+ur)]+(a+ur)-a-ur, & \text{and } U_\gamma\cap S(k)\neq\emptyset,\ \text{where } k\in\{1,2\}; \\ \pi(\gamma)-|U_\gamma\cap S(1)|uwz_1-|U_\gamma\cap S(2)|uwz_2, & \text{otherwise.} \end{cases} \tag{F.11}$$

$$= \pi(\gamma)-[(uwz_1)\wedge(a+ur)]|U_\gamma\cap S(1)|-[(uwz_2)\wedge(a+ur)]|U_\gamma\cap S(2)| \tag{F.12}$$

which yields (F.1). □

Now instead of auditing every remaining supplier in the RP subphase the buyer distinguishes between tier-1 and tier-2 suppliers and make auditing decisions on them separately. The following replaces Theorem 3:

THEOREM F.1. *There exists an optimal policy $\xi^*\in\Xi$ with the property that auditing decisions are divided into two subphases:*

(a) AD subphase: *To audit and drop (* AD*) some suppliers (or no supplier); followed by*

(b) RP subphase:

i. *If $a+ur\leqslant uwz_1$ and $a+ur\leqslant uwz_2$, audit and rectify (*AR*) all remaining unaudited suppliers in any sequence, then proceed to production (*PP*);*

ii. *If $a+ur\leqslant uwz_1$ and $a+ur>uwz_2$, audit and rectify (*AR*) all remaining unaudited suppliers in tier 1 in any sequence, then proceed to production (*PP*);*

iii. *If $a+ur>uwz_1$ and $a+ur\leqslant uwz_2$, audit and rectify (*AR*) all remaining unaudited suppliers in tier 2 in any sequence, then proceed to production (*PP*);*

iv. *If $a+ur>uwz_1$ and $a+ur>uwz_2$, proceed to production directly (*PP*).*

Proof. The result is a direct consequence of Propositions D.1 and F.1. □

COROLLARY F.1. *At state $\gamma\in\Gamma$ if the optimal policy ξ^* is already in the* RP *subphase,*

$$V^*(\gamma)=\pi(\gamma)-[(uwz_1)\wedge(a+ur)]|U_\gamma\cap S(1)|-[(uwz_2)\wedge(a+ur)]|U_\gamma\cap S(2)|. \tag{F.13}$$

Proof. The result follows (F.1) in Proposition F.1. □

F.2. Optimal Auditing Sequence in Tier 2

Since Theorem 4 only has to do with tier-2 firms, it remains the same other than now z is replaced with z_2. Corollary 2 remains the same since it does not involve penalty.

F.3. Supplier Choice When Auditing One Firm

Assume $\delta \equiv z_1 - z_2 \geqslant 0$. We consider a state γ where $U_\gamma^{(1)} \neq \emptyset$ and $U_\gamma^{(2)} \neq \emptyset$ (otherwise obviously all results stay the same by replacing z with z_k so that tier k has unaudited suppliers). We also exclude the state $\gamma_\diamond = (g_\diamond, U_\diamond)$ ("$\diamond$" read "diamond") where $g_\diamond = (\{\text{A},\text{B}\}, \emptyset, \emptyset, \{1\})$ and $U_\diamond = \{\text{A},\text{B},1\}$, which creates complications, not insights, due to the sole tier-2 supplier dominating the network claiming both tier-1 firms dependents.

Proposition F.2. *Consider a nonterminal state $\gamma \neq \gamma_\diamond$ in which $U_\gamma^{(1)} \neq \emptyset$ and $U_\gamma^{(2)} \neq \emptyset$. Suppose the buyer can audit at most one supplier before proceeding to production. As δ increases, the optimal action shifts from proceeding to production (*PP*) to auditing and dropping (*AD*) a tier-2 supplier supplier, then either to auditing and dropping (*AD*) a tier-1 supplier or to auditing and rectifying (*AR*) a tier-1 supplier.*

Proof. We first calculate the value from each action for comparison later. Let ξ_{PP} be the policy that prescribes PP for every state (i.e., $\xi_{\text{PP}}(\gamma) = \text{PP}, \forall \gamma \in \Gamma$). For $i \in U_\gamma^{(1)}$ and $j \in U_\gamma^{(2)}$,

$$\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) = -a + (1-u)V(\xi_{\text{PP}}, \gamma \oplus i) + uV(\xi_{\text{PP}}, \gamma \ominus i) \tag{F.14}$$

$$= -a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1)] \tag{F.15}$$

$$+ u[\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}|] \tag{F.16}$$

$$\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j)) = -a + (1-u)V(\xi_{\text{PP}}, \gamma \oplus j) + uV(\xi_{\text{PP}}, \gamma \ominus j) \tag{F.17}$$

$$= -a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}|] \tag{F.18}$$

$$+ u[\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}|] \tag{F.19}$$

$$\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i)) = -a - ur + V(\xi_{\text{PP}}, \gamma \oplus i) \tag{F.20}$$

$$= -a - ur + \pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1) \tag{F.21}$$

$$\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(j)) = -a - ur + V(\xi_{\text{PP}}, \gamma \oplus j) \tag{F.22}$$

$$= -a - ur + \pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}| \tag{F.23}$$

$$\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{PP}) = \pi(\gamma) - \zeta(\gamma) \tag{F.24}$$

$$= \pi(\gamma) - uwz_2|U_\gamma| - uw\delta|U_\gamma^{(1)}|. \tag{F.25}$$

By (F.21) and (F.23), $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(j))$, so $\text{AR}(i)$ always dominates $\text{AR}(j)$ (weakly if $\delta = 0$). Therefore we do not consider $\text{AR}(j)$ as a candidate for the optimal action.

We proceed by characterizing the conditions for each action to be optimal. We then identify the patterns Proposition F.2 describes.

Condition for PP *to be optimal.* By (F.25) and (F.21) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{PP}) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i))$ where $i \in U_\gamma^{(1)}$ if and only if

$$\delta \leqslant \frac{a - ur}{uw} - z_2. \tag{F.26}$$

By (F.25) and (F.16) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{PP}) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i))$ where $i \in U_\gamma^{(1)}$ if and only if

$$\pi(\gamma) - uwz_2|U_\gamma| - uw\delta|U_\gamma^{(1)}| \geqslant$$
$$-a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1)] + u[\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}|] \tag{F.27}$$

which, by rearranging terms and noting that a tier-1 firm cannot have a tier-1 dependent so that $|U^{(1)}_{\gamma\ominus i}| = |U^{(1)}_{\gamma}| - 1$, is equivalent to

$$\begin{aligned}\pi(\gamma) - uwz_2|U_\gamma| - uw\delta|U^{(1)}_\gamma| \geqslant\\ -a + [\pi(\gamma) - uwz_2|U_\gamma| - uw\delta|U^{(1)}_\gamma|] + uwz_2 + uw\delta - u[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U^{(1)}_\gamma| - 1)]\\ + u[\pi(\gamma\ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta(|U^{(1)}_\gamma| - 1)] \quad \text{(F.28)}\end{aligned}$$

or

$$a - uwz_2 + u[\nabla(\gamma,i) - uwz_2(|U_\gamma| - |U_{\gamma\ominus i}| - 1)] \geqslant uw\delta \quad \text{(F.29)}$$

or

$$\delta \leqslant \frac{a + u\{\nabla(\gamma,i) - wz_2[1 + u(|U_\gamma| - |U_{\gamma\ominus i}| - 1)]\}}{uw} \equiv \delta^{(1)}_d(\gamma,i). \quad \text{(F.30)}$$

By (F.25) and (F.19) $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP}) \geqslant \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(j))$ where $j \in U^{(2)}_\gamma$ if and only if

$$\begin{aligned}\pi(\gamma) - uwz_2|U_\gamma| - uw\delta|U^{(1)}_\gamma| \geqslant\\ -a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U^{(1)}_\gamma|] + u[\pi(\gamma\ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U^{(1)}_{\gamma\ominus j}|] \quad \text{(F.31)}\end{aligned}$$

or

$$\begin{aligned}\pi(\gamma) - uwz_2|U_\gamma| - uw\delta|U^{(1)}_\gamma| \geqslant\\ -a + [\pi(\gamma) - uwz_2(|U_\gamma|) - uw\delta|U^{(1)}_\gamma|] + uwz_2 - u[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U^{(1)}_\gamma|]\\ + u[\pi(\gamma\ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U^{(1)}_{\gamma\ominus j}|] \quad \text{(F.32)}\end{aligned}$$

or

$$a + u\{\nabla(\gamma,j) - wz_2[1 + u(|U_\gamma| - |U_{\gamma\ominus j}| - 1)] - uw\delta(|U^{(1)}_\gamma| - |U^{(1)}_{\gamma\ominus j}|)\} \geqslant 0. \quad \text{(F.33)}$$

For j without an unaudited tier-1 dependent, i.e., $|U^{(1)}_\gamma| - |U^{(1)}_{\gamma\ominus j}| = 0$, (F.33) is equivalent to

$$a + u\{\nabla(\gamma,j) - wz_2[1 + u(|U_\gamma| - |U_{\gamma\ominus j}| - 1)]\} \geqslant 0. \quad \text{(F.34)}$$

Since $|U_\gamma| - |U_{\gamma\ominus j}| \geqslant 1$, $1 + u(|U_\gamma| - |U_{\gamma\ominus j}| - 1) > 0$, which implies (F.34) is equivalent to

$$z_2 \leqslant \frac{a + u\nabla(\gamma,j)}{uw[1 + u(|U_\gamma| - |U_{\gamma\ominus j}| - 1)]} \equiv z^{(2)}_d(\gamma,j). \quad \text{(F.35)}$$

For j with an unaudited tier-1 dependent, i.e., $|U^{(1)}_\gamma| - |U^{(1)}_{\gamma\ominus j}| > 0$, (F.33) is equivalent to

$$\delta \leqslant \frac{a + u\{\nabla(\gamma,j) - wz_2[1 + u(|U_\gamma| - |U_{\gamma\ominus j}| - 1)]\}}{uw(|U^{(1)}_\gamma| - |U^{(1)}_{\gamma\ominus j}|)} \equiv \delta^{(2)}_d(\gamma,j). \quad \text{(F.36)}$$

Therefore PP is optimal if and only if

- $\delta \leqslant \frac{a-ur}{uw} - z_2$;
- $\delta \leqslant \delta^{(1)}_d(\gamma,i)$ for every $i \in U^{(1)}_\gamma$;
- $z_2 \leqslant z^{(2)}_d(\gamma,j)$ for every $j \in U^{(2)}_\gamma$ without an unaudited tier-1 dependent; and $\delta \leqslant \delta^{(2)}_d(\gamma,j)$ for every $j \in U^{(2)}_\gamma$ with an unaudited tier-1 dependent.

Condition for AR(i) *for any* $i \in U_\gamma^{(1)}$ *to be optimal.* By (F.26) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{PP})$ if and only if

$$\delta > \frac{a - ur}{uw} - z_2. \tag{F.37}$$

Let $i' \in U_\gamma^{(1)}$. By (F.21) and (F.16) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i'))$ if and only if

$$\begin{aligned} &-a - ur + \pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}| \geqslant \\ &\quad -a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1)] + u[\pi(\gamma \ominus i') - uwz_2|U_{\gamma\ominus i'}| - uw\delta|U_{\gamma\ominus i'}^{(1)}|] \end{aligned} \tag{F.38}$$

or

$$-ur \geqslant -u[\nabla(\gamma, i') - uwz_2(|U_\gamma| - |U_{\gamma\ominus i'}| - 1) - uw\delta(|U_\gamma^{(1)}| - |U_{\gamma\ominus i'}^{(1)}| - 1)]. \tag{F.39}$$

which, since $|U_\gamma^{(1)}| - |U_{\gamma\ominus i'}^{(1)}| = 1$, is equivalent to

$$-r + \nabla(\gamma, i') \geqslant uwz_2(|U_\gamma| - |U_{\gamma\ominus i'}| - 1). \tag{F.40}$$

For i' with an unaudited tier-2 dependent, i.e., $|U_\gamma| - |U_{\gamma\ominus i'}| - 1 > 0$, (F.40) is equivalent to

$$z_2 \leqslant \frac{-r + \nabla(\gamma, i')}{uw(|U_\gamma| - |U_{\gamma\ominus i'}| - 1)} \equiv z_r^{(1)}(\gamma, i'). \tag{F.41}$$

For i' without an unaudited tier-2 dependent, i.e., $|U_\gamma| - |U_{\gamma\ominus i'}| - 1 = 0$, (F.40) is equivalent to

$$\nabla(\gamma, i') \geqslant r. \tag{F.42}$$

Let $j \in U_\gamma^{(2)}$. By (F.21) and (F.19) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j))$ if and only if

$$\begin{aligned} &-a - ur + \pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1) \geqslant \\ &\quad -a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}|] + u[\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}|] \end{aligned} \tag{F.43}$$

or

$$-ur + uw\delta \geqslant -u[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}|] + u[\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}|] \tag{F.44}$$

or

$$-r + w\delta \geqslant -[\nabla(\gamma, j) - uwz_2(|U_\gamma| - |U_{\gamma\ominus j}| - 1) - uw\delta(|U_\gamma^{(1)}| - |U_{\gamma\ominus j}^{(1)}|)] \tag{F.45}$$

or

$$w\delta[u(|U_\gamma^{(1)}| - |U_{\gamma\ominus j}^{(1)}|) - 1] \leqslant -r + \nabla(\gamma, j) - uwz_2(|U_\gamma| - |U_{\gamma\ominus j}| - 1). \tag{F.46}$$

$|U_\gamma^{(1)}| - |U_{\gamma\ominus j}^{(1)}|$ is the number of tier-1 dependents of tier-2 supplier j—either 0 or 1 given that $\gamma \neq \gamma_\diamond$. Since $u < 1$, this implies $u(|U_\gamma^{(1)}| - |U_{\gamma\ominus j}^{(1)}|) - 1 < 0$. Hence (F.46) is equivalent to

$$\delta \geqslant \frac{-r + \nabla(\gamma, j) - uwz_2(|U_\gamma| - |U_{\gamma\ominus j}| - 1)}{w[u(|U_\gamma^{(1)}| - |U_{\gamma\ominus j}^{(1)}|) - 1]} \equiv \delta_r(\gamma, j). \tag{F.47}$$

Therefore AR(i) is optimal if and only if

- $\delta > \frac{a-ur}{uw} - z_2$;
- $z_2 \leqslant z_r^{(1)}(\gamma, i')$ for any $i' \in U_\gamma^{(1)}$ with an unaudited tier-2 dependent; $\nabla(\gamma, i') \geqslant r$ for any $i' \in U_\gamma^{(1)}$ without an unaudited tier-2 dependent; and
- $\delta \geqslant \delta_r(\gamma, j)$ for any $j \in U_\gamma^{(2)}$.

Condition for AD(i) *where* $i \in U_\gamma^{(1)}$ *to be optimal.* By (F.30) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{PP})$ if and only if $\delta > \delta_d^{(1)}(\gamma, i)$.

Let $i' \in U_\gamma^{(1)}$. By (F.41), in the case where i has an unaudited tier-2 dependent, $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i'))$ if and only if $z_2 > z_r^{(1)}(\gamma, i)$. By (F.42), in the case where i does not have an unaudited tier-2 dependent, $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i'))$ if and only if $\nabla(\gamma, i) < r$.

By (F.16) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i'))$ if and only if

$$\begin{aligned} &-a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1)] + u[\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}|] \geqslant \\ &\quad -a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1)] + u[\pi(\gamma \ominus i') - uwz_2|U_{\gamma\ominus i'}| - uw\delta|U_{\gamma\ominus i'}^{(1)}|] \end{aligned} \tag{F.48}$$

or

$$u[\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}|] \geqslant u[\pi(\gamma \ominus i') - uwz_2|U_{\gamma\ominus i'}| - uw\delta|U_{\gamma\ominus i'}^{(1)}|] \tag{F.49}$$

or

$$\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}| \geqslant \pi(\gamma \ominus i') - uwz_2|U_{\gamma\ominus i'}| - uw\delta|U_{\gamma\ominus i'}^{(1)}|. \tag{F.50}$$

Since the only tier-1 dependent of i or i' is itself, $|U_{\gamma\ominus i}^{(1)}| = |U_{\gamma\ominus i'}^{(1)}|$. Hence (F.50) is equivalent to

$$\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| \geqslant \pi(\gamma \ominus i') - uwz_2|U_{\gamma\ominus i'}|. \tag{F.51}$$

Let $j \in U_\gamma^{(2)}$. By (F.16) and (F.19) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j))$ if and only if

$$\begin{aligned} &-a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta(|U_\gamma^{(1)}| - 1)] + u[\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}|] \\ &\quad \geqslant -a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}|] + u[\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}|] \end{aligned} \tag{F.52}$$

or

$$(1-u)uw\delta + u[\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}|] \geqslant u[\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}|] \tag{F.53}$$

or

$$(1-u)w\delta + \pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| - uw\delta|U_{\gamma\ominus i}^{(1)}| \geqslant \pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}| \tag{F.54}$$

or

$$w\delta[(1-u) + u(|U_{\gamma\ominus j}^{(1)}| - |U_{\gamma\ominus i}^{(1)}|)] \geqslant \pi(\gamma \ominus j) - \pi(\gamma \ominus i) - uwz_2(|U_{\gamma\ominus j}| - |U_{\gamma\ominus i}|). \tag{F.55}$$

Since $\gamma \neq \gamma_\diamond$, $|U_{\gamma\ominus j}^{(1)}| - |U_{\gamma\ominus i}^{(1)}| \geqslant 0$. Hence (F.55) is equivalent to

$$\delta \geqslant \frac{\pi(\gamma \ominus j) - \pi(\gamma \ominus i) - uwz_2(|U_{\gamma\ominus j}| - |U_{\gamma\ominus i}|)}{w[1 + u(|U_{\gamma\ominus j}^{(1)}| - |U_{\gamma\ominus i}^{(1)}| - 1)]} \equiv \delta_d(\gamma, i, j). \tag{F.56}$$

Therefore AD(i) is optimal if and only if

- $\delta > \delta_d^{(1)}(\gamma, i)$;
- In the case where i has an unaudited tier-2 dependent, $z_2 > z_r^{(1)}(\gamma, i)$; in the case where i does not have an unaudited tier-2 dependent, $\nabla(\gamma, i) < r$;
- $\pi(\gamma \ominus i) - uwz_2|U_{\gamma\ominus i}| \geqslant \pi(\gamma \ominus i') - uwz_2|U_{\gamma\ominus i'}|$ for $i' \in U_\gamma^{(1)}$; and
- $\delta \geqslant \delta_d(\gamma, i, j)$ for every $j \in U_\gamma^{(2)}$.

Condition for AD(j) *where* $j \in U_\gamma^{(2)}$ *to be optimal.* By (F.35), in the case where j does not have an unaudited tier-1 dependent, $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{PP})$ if and only if $z_2 > z_d^{(2)}(\gamma, j)$. By (F.36), in the case where j has an unaudited tier-1 dependent, $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{PP})$ if and only if $\delta > \delta_d^{(2)}(\gamma, j)$.

Let $i \in U_\gamma^{(1)}$. By (F.47) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AR}(i))$ if and only if $\delta < \delta_r(\gamma, j)$.

By (F.56) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j)) > \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i))$ if and only if $\delta < \delta_d(\gamma, i, j)$.

Let $j' \in U_\gamma^{(2)}$. By (F.19) $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(j'))$ if and only if

$$-a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}|] + u[\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}|] \geqslant$$
$$-a + (1-u)[\pi(\gamma) - uwz_2(|U_\gamma| - 1) - uw\delta|U_\gamma^{(1)}|] + u[\pi(\gamma \ominus j') - uwz_2|U_{\gamma\ominus j'}| - uw\delta|U_{\gamma\ominus j'}^{(1)}|] \quad \text{(F.57)}$$

or

$$\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}| \geqslant \pi(\gamma \ominus j') - uwz_2|U_{\gamma\ominus j'}| - uw\delta|U_{\gamma\ominus j'}^{(1)}|. \quad \text{(F.58)}$$

Therefore AD(j) is optimal if and only if

- In the case where j does not have an unaudited tier-1 dependent, $z_2 > z_d^{(2)}(\gamma, j)$; in the case where j has an unaudited tier-1 dependent, $\delta > \delta_d^{(2)}(\gamma, j)$;
- $\delta < \delta_r(\gamma, j)$;
- $\delta < \delta_d(\gamma, i, j)$ for every $i \in U_\gamma^{(1)}$; and
- $\pi(\gamma \ominus j) - uwz_2|U_{\gamma\ominus j}| - uw\delta|U_{\gamma\ominus j}^{(1)}| \geqslant \pi(\gamma \ominus j') - uwz_2|U_{\gamma\ominus j'}| - uw\delta|U_{\gamma\ominus j'}^{(1)}|$ for every $j' \in U_\gamma^{(2)}$.

Given the above necessary and sufficient condition for each action to be optimal, we make the following observations about how the buyer's optimal action changes as δ varies:

(*a*) Either PP does not arise as an optimal action for any value of δ (if there is $j \in U_\gamma^{(2)}$ with an unaudited tier-1 dependent such that $z_2 \leqslant z_d^{(2)}(\gamma, j)$) or PP is optimal for all sufficiently low values of δ (specifically $\delta \leqslant \frac{a-ur}{uw} - z_2$, $\delta \leqslant \delta_d^{(1)}(\gamma, i)$ for every $i \in U_\gamma^{(1)}$, and $\delta \leqslant \delta_d^{(2)}(\gamma, j)$ for every $j \in U_\gamma^{(2)}$ with an unaudited tier-1 dependent).

(*b*) Any value of δ that makes AD(j) for some unaudited tier-2 supplier j optimal is greater than any value of δ that makes PP optimal. (If j does not have an unaudited tier-1 supplier and $z_2 > z_d^{(2)}(\gamma, j)$, PP does not arise as optimal for any δ; if j does not have an unaudited tier-1 supplier but $z_2 \leqslant z_d^{(2)}(\gamma, j)$, AD$(j)$ does not arise as optimal for any δ; otherwise if j has an unaudited tier-1 supplier then AD(j) being optimal requires $\delta > \delta_d^{(2)}(\gamma, j)$ yet PP being optimal requires $\delta \leqslant \delta_d^{(2)}(\gamma, j)$.)

(*c*) Any value of δ that makes AR(i) for some unaudited tier-1 supplier i optimal is greater than any value of δ that makes AD(j) for some unaudited tier-2 supplier j optimal. (AR(i) being optimal requires $\delta \geqslant \delta_r(\gamma, j)$ while AD(j) being optimal requires $\delta < \delta_r(\gamma, j)$.)

(*d*) Any value of δ that makes AD(i) for some unaudited tier-1 supplier i optimal is greater than any value of δ that makes AD(j) for some unaudited tier-2 supplier j optimal. (AD(i) being optimal requires $\delta \geqslant \delta_d(\gamma, i, j)$ while AD(j) being optimal requires $\delta < \delta_d(\gamma, i, j)$.)

(*e*) If the parameters make AD(i) for some unaudited tier-1 supplier i optimal, varying δ alone will never make AR(i') optimal for any unaudited tier-1 supplier i'. (If i has an unaudited tier-2 dependent, AD(i) being optimal implies $z_2 > z_r^{(1)}(\gamma, i)$—a condition independent of δ—but AR(i') being optimal requires the opposite, i.e., $z_2 \leqslant z_r^{(1)}(\gamma, i)$; if i does not have an unaudited tier-2 dependent, AD(i) being optimal implies $\nabla(\gamma, i) < r$—a condition independent of δ—but AR(i') being optimal requires the opposite, i.e., $\nabla(\gamma, i) \geqslant r$.) The above five observations together imply the pattern described in Proposition F.2. □

Appendix G: Results with Inaccurate Detection in Auditing

We consider an extension in which the audit is not accurate in the sense that it sometimes may not detect an existing noncompliance at a supplier. (While an audit may fail to uncover a real violation at a noncompliant supplier, it will never falsely identify a violation at a compliant supplier.) Let there be two types of noncompliant suppliers: one type whose noncompliance will be detected by the audit and one type whose noncompliance will escape the audit. Let d be the probability that a noncompliant supplier is of the former type so the noncompliance will be detected by the audit. Inaccurate detection leads to the possibility of a supplier being noncompliant even having passed an audit. Specifically the probability that a supplier is still noncompliant after it has passed an audit is

$$\mathbb{P}\{\text{noncompliant}|\text{passed audit}\} = \frac{\mathbb{P}\{\text{noncompliant and passed audit}\}}{\mathbb{P}\{\text{passed audit}\}} = \frac{u(1-d)}{(1-u)+u(1-d)} \equiv \widehat{u}. \tag{G.1}$$

It is easy to verify that $u \leqslant 1$ implies $\widehat{u} \leqslant u$. We assume a supplier that has undergone a rectification process has the same probability of noncompliance $\widehat{u}$ as a supplier that has passed an audit. (In practice a buyer often deems a supplier's rectification complete once it passes a followup audit.) Therefore in this extension each vetted supplier has the same probability of noncompliance $\widehat{u}$. The expected total penalty from violations on state $\gamma = (g, U)$ is

$$\zeta(\gamma) = wz(\widehat{u}|S_g \backslash U| + u|U|) = wz[\widehat{u}|S_g| + (u-\widehat{u})|U|]. \tag{G.2}$$

Note $|S_g \backslash U|$ is the number of vetted suppliers in γ.

Only the following formulae of expected values differ from the base model: given auditing policy $\xi \in \Xi$ and state $\gamma \in \Gamma$, and $i \in U_\gamma$,

$$\widetilde{V}(\xi, \gamma, \text{AD}(i)) = -a + (1-ud)V(\xi, \gamma \oplus i) + udV(\xi, \gamma \ominus i). \tag{G.3}$$

$$\widetilde{V}(\xi, \gamma, \text{AR}(i)) = -a + (1-ud)V(\xi, \gamma \oplus i) + ud(-r + V(\xi, \gamma \oplus i)) \tag{G.4}$$

$$= -a - udr + V(\xi, \gamma \oplus i) \tag{G.5}$$

and similarly

$$\widetilde{V}^*(\gamma, \text{AD}(i)) = -a + (1-ud)V^*(\gamma \oplus i) + udV^*(\gamma \ominus i) \tag{G.6}$$

$$\widetilde{V}^*(\gamma, \text{AR}(i)) = -a + (1-ud)V^*(\gamma \oplus i) + ud(-r + V^*(\gamma \oplus i)) \tag{G.7}$$

$$= -a - udr + V^*(\gamma \oplus i). \tag{G.8}$$

G.1. The Two Subphases

PROPOSITION G.1. *The buyer can be at least as well off by postponing all audit and rectify (* AR*) actions to after all audit and drop (* AD*) actions.*

Proof. Let $\xi \in \Xi$ be such that there exists $\gamma = (g, U) \in \Gamma$, $i \in U_\gamma$, and $j \in U_{\gamma \oplus i}$ such that

$$\xi(\gamma) = \text{AR}(i) \quad \text{and} \quad \xi(\gamma \oplus i) = \text{AD}(j). \tag{G.9}$$

(If there does not exist such a triple of γ, i, and j then in ξ already all AR actions come after all AD actions.) We specify a policy $\xi' \in \Xi$ otherwise identical to ξ but with the sequence of the above two actions swapped, namely,

$$\xi'(\gamma) = \text{AD}(j) \tag{G.10}$$

$$\xi'(\gamma \oplus j) = \text{AR}(i) \tag{G.11}$$

$$\xi'(\gamma \ominus j) = \begin{cases} \text{AR}(i), & \text{if } i \notin D_g(j) \\ \xi(\gamma \ominus j), & \text{if } i \in D_g(j) \end{cases} \tag{G.12}$$

$$\xi'(\gamma') = \xi(\gamma'), \quad \forall \gamma' \in \Gamma \backslash \{\gamma, \gamma \oplus j, \gamma \ominus j\}. \tag{G.13}$$

It suffices to show $V(\xi', \gamma) \geqslant V(\xi, \gamma)$.

Now

$$\begin{aligned} V(\xi,\gamma) &= \widetilde{V}(\xi,\gamma,\text{AR}(i)) && \text{(G.14)} \\ &= -a - udr + V(\xi, \gamma \oplus i) && \text{(G.15)} \\ &= -a - udr + \widetilde{V}(\xi, \gamma \oplus i, \text{AD}(j)) \qquad \text{(by (G.9))} && \text{(G.16)} \\ &= -a - udr - a + (1-ud)V(\xi, \gamma \oplus i \oplus j) + uV(\xi, \gamma \oplus i \ominus j) \qquad \text{(by (G.3))} && \text{(G.17)} \end{aligned}$$

and

$$\begin{aligned} V(\xi',\gamma) &= \widetilde{V}(\xi',\gamma,\text{AD}(j)) && \text{(G.18)} \\ &= -a + (1-ud)V(\xi', \gamma \oplus j) + udV(\xi', \gamma \ominus j). && \text{(G.19)} \end{aligned}$$

There are two cases of i:

- *Case 1:* $i \notin D_g(j)$. Then

$$\begin{aligned} V(\xi',\gamma) &= -a + (1-ud)\widetilde{V}(\xi', \gamma \oplus j, \text{AR}(i)) + ud\widetilde{V}(\xi', \gamma \ominus j, \text{AR}(i)) && \text{(G.20)} \\ &= -a + (1-ud)(-a - udr + V(\xi', \gamma \oplus j \oplus i)) \\ &\quad + ud(-a - udr + V(\xi', \gamma \ominus j \oplus i)) && \text{(G.21)} \\ &= -a - a - udr + (1-ud)V(\xi', \gamma \oplus j \oplus i) + udV(\xi', \gamma \ominus j \oplus i) && \text{(G.22)} \end{aligned}$$

Note that $\xi'|_{R^+(\gamma \oplus j \oplus i)} = \xi|_{R^+(\gamma \oplus i \oplus j)}$, so $V(\xi', \gamma \oplus j \oplus i) = V(\xi, \gamma \oplus i \oplus j)$. Since $i \notin D_g(j)$, $\gamma \ominus j \oplus i = \gamma \oplus i \ominus j$. Also, $\xi'|_{R^+(\gamma \ominus j \oplus i)} = \xi|_{R^+(\gamma \oplus i \ominus j)}$. Hence, $V(\xi', \gamma \ominus j \oplus i) = V(\xi, \gamma \oplus i \ominus j)$. Therefore by comparing (G.17) and (G.22) we conclude $V(\xi', \gamma) = V(\xi, \gamma)$.

- *Case 2:* $i \in D_g(j)$. Immediately, $\gamma \oplus i \ominus j = \gamma \ominus j$.

$$\begin{aligned} V(\xi',\gamma) &= -a + (1-ud)\widetilde{V}(\xi', \gamma \oplus j, \text{AR}(i)) + udV(\xi', \gamma \ominus j) && \text{(G.23)} \\ &= -a + (1-ud)(-a - udr + V(\xi', \gamma \oplus j \oplus i)) + udV(\xi', \gamma \ominus j) && \text{(G.24)} \\ &= -a - (1-ud)(a + udr) + (1-ud)V(\xi', \gamma \oplus j \oplus i) + udV(\xi', \gamma \ominus j). && \text{(G.25)} \end{aligned}$$

Same as above, since $\xi'|_{R^+(\gamma \oplus j \oplus i)} = \xi|_{R^+(\gamma \oplus i \oplus j)}$,

$$V(\xi', \gamma \oplus j \oplus i) = V(\xi, \gamma \oplus i \oplus j). \tag{G.26}$$

Since $i \in D_g(j)$, $\xi'(\gamma \ominus j) = \xi(\gamma \ominus j)$. Hence $\xi'|_{R^+(\gamma \ominus j)} = \xi|_{R^+(\gamma \ominus j)}$. By $\gamma \oplus i \ominus j = \gamma \ominus j$,

$$V(\xi, \gamma \oplus i \ominus j) = V(\xi, \gamma \ominus j) = V(\xi', \gamma \ominus j). \tag{G.27}$$

Substitute (G.26) and (G.27) into (G.25), then subtract (G.17) to obtain

$$V(\xi', \gamma) - V(\xi, \gamma) = a + udr - (1 - ud)(a + udr) = ud(a + udr) \geqslant 0. \tag{G.28}$$

This completes the proof. Note that the buyer is strictly better off by swapping the actions $\text{AR}(i)$ and $\text{AD}(j)$ (as in ξ') if and only if $i \in D_g(j)$ and $ud(a + udr) > 0$. □

PROPOSITION G.2. *Limit the buyer's actions to audit and rectify (* AR*) unaudited suppliers and proceed to production (* PP*). The optimal auditing policy is to audit and rectify all unaudited suppliers in any sequence if $a + udr \leqslant (u - \widehat{u})wz$ and to proceed to production if $a + udr \geqslant (u - \widehat{u})wz$. Furthermore, given $\gamma \in \Gamma$,*

$$V^*(\gamma) = \pi(\gamma) - \widehat{u}wz|S_{g_\gamma}| - \{(a + udr) \wedge [(u - \widehat{u})wz]\}|U_\gamma|. \tag{G.29}$$

Proof. Given $\gamma \in \Gamma$ and $i \in U_\gamma$, by (2) and the definition of (G.2),

$$\widetilde{V}^*(\gamma, \text{PP}) = \pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| - (u - \widehat{u})|U_\gamma|]. \tag{G.30}$$

We prove the result by mathematical induction on the number of unaudited supplier in the state, $|U_\gamma|$. If $|U_\gamma| = 1$, let $i \in U_\gamma$, then $\gamma \oplus i$ is a terminal state. By (G.8),

$$\widetilde{V}^*(\gamma, \text{AR}(i)) = -a - udr + V^*(\gamma \oplus i) = -a - udr + \pi(\gamma \oplus i) - \widehat{u}wz|S_{g_\gamma}| = -a - udr + \pi(\gamma) - \widehat{u}wz|S_{g_\gamma}| \tag{G.31}$$

since states $\gamma \oplus i$ and γ have the same underlying supply network, which determines the production profit. Note $\widetilde{V}^*(\gamma, \text{AR}(i))$ is independent of i. $\text{AR}(i)$ is preferred to PP iff $\widetilde{V}^*(\gamma, \text{AR}(i)) > \widetilde{V}^*(\gamma, \text{PP})$, or (since $|U_\gamma| = 1$)

$$-a - udr + \pi(\gamma) - \widehat{u}wz|S_{g_\gamma}| > \pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| - (u - \widehat{u})] \tag{G.32}$$

or $a + udr < (u - \widehat{u})wz$. Hence,

$$V^*(\gamma) = \widetilde{V}^*(\gamma, \text{AR}(i)) \vee \widetilde{V}^*(\gamma, \text{PP}) = \pi(\gamma) - |S_{g_\gamma}|(\widehat{u}wz) - (a + udr) \wedge [(u - \widehat{u})wz]. \tag{G.33}$$

By mathematical induction, suppose given $m \in \mathbb{N}^+$, for any $\gamma' \in \Gamma$ such that $|U_{\gamma'}| = m$,

$$V^*(\gamma') = \pi(\gamma') - \widehat{u}wz|S_{g_{\gamma'}}| - m\{(a + udr) \wedge [(u - \widehat{u})wz]\}. \tag{G.34}$$

Now let $\gamma \in \Gamma$ be such that $|U_\gamma| = m + 1$. Pick arbitrary $i \in U_\gamma$, then $|U_{\gamma \oplus i}| = m$. By (G.8) and (G.34),

$$\widetilde{V}^*(\gamma, \text{AR}(i)) = -a - udr + V^*(\gamma \oplus i) \tag{G.35}$$

$$= -a - udr + \pi(\gamma) - \widehat{u}wz|S_{g_\gamma}| - m\{(a + udr) \wedge [(u - \widehat{u})wz]\}. \tag{G.36}$$

$\text{AR}(i)$ is preferred to PP iff $\widetilde{V}^*(\gamma, \text{AR}(i)) > \widetilde{V}^*(\gamma, \text{PP})$, or

$$-a - udr + \pi(\gamma) - \widehat{u}wz|S_{g_\gamma}| - m\{(a + udr) \wedge [(u - \widehat{u})wz]\} > \pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| + (m + 1)(u - \widehat{u})] \tag{G.37}$$

or

$$a + udr + m\{(a + udr) \wedge [(u - \widehat{u})wz]\} < wz(m + 1)(u - \widehat{u}) \tag{G.38}$$

which holds if and only if $a+udr<wz(u-\widehat{u})$, as we wanted to show. Finally, to complete the induction step, note

$$V^*(\gamma)=\begin{cases}\widetilde{V}^*(\gamma,\text{PP}), & \text{if } a+udr\geqslant wz(u-\widehat{u})\\ \widetilde{V}^*(\gamma,\text{AR}(i)), & \text{if } a+udr<wz(u-\widehat{u})\end{cases} \tag{G.39}$$

$$=\begin{cases}\pi(\gamma)-\widehat{u}wz|S_{g_\gamma}|-(m+1)(u-\widehat{u})wz, & \text{if } a+udr\geqslant wz(u-\widehat{u})\\ -a-udr+\pi(\gamma)-\widehat{u}wz|S_{g_\gamma}|-m\{(a+udr)\wedge[(u-\widehat{u})wz]\}, & \text{if } a+udr<wz(u-\widehat{u})\end{cases} \tag{G.40}$$

$$=\begin{cases}\pi(\gamma)-\widehat{u}wz|S_{g_\gamma}|-(m+1)(u-\widehat{u})wz, & \text{if } a+udr\geqslant wz(u-\widehat{u})\\ \pi(\gamma)-\widehat{u}wz|S_{g_\gamma}|-(m+1)(a+udr), & \text{if } a+udr<wz(u-\widehat{u})\end{cases} \tag{G.41}$$

$$=\pi(\gamma)-\widehat{u}wz|S_{g_\gamma}|-(m+1)\{(a+udr)\wedge[(u-\widehat{u})wz]\}. \quad \square \tag{G.42}$$

THEOREM G.1. *There exists an optimal policy $\xi^*\in\Xi$ with the property that auditing decisions are divided into two subphases:*

(*a*) AD subphase: *To audit and drop (*AD*) some suppliers (or no supplier); followed by*

(*b*) RP subphase: *To audit and rectify (*AR*) all remaining unaudited suppliers in an arbitrary sequence if $a+udr<wz(u-\widehat{u})$; or to proceed to production (*PP*) if $a+udr\geqslant wz(u-\widehat{u})$.*

Proof. The result is a direct consequence of Propositions G.1 and G.2. $\square$

We denote the additional cost associated with each unaudited supplier (relative to a vetted supplier) in the RP subphase $\widehat{c}_{\text{RP}}\equiv(a+udr)\wedge[(u-\widehat{u})wz]$.

COROLLARY G.1. *At state $\gamma\in\Gamma$, if the optimal policy ξ^* is already in the* RP *subphase,*

$$V^*(\gamma)=\pi(\gamma)-\widehat{u}wz|S_{g_\gamma}|-\widehat{c}_{\text{RP}}|U_\gamma|. \tag{G.43}$$

Proof. The result follows (G.29) in Proposition G.2. $\square$

G.2. Optimal Auditing Sequence

When we focus on tier 2, a problem that arises with inaccurate detection is that the dropping of a tier-2 supplier (call it i) may cause another supplier (i') to carry additional (vetted) dependents (in tier 1) compared to before the dropping of i. Even if supplier i' was not favored for an audit in an earlier state, it may become favorable in a subsequent step. (We can view this as the breaking down of the decreasing differences property when we augment the production profit with the "residual penalty" of vetted suppliers arising from $\widehat{u}$: supplier i' may become less valuable (accounting for $\widehat{u}$) after the dropping of i.)

We propose a fix here (only for Section G.2) that imposes perfect compliance on the tier-1 firms: that is, they are not at risk of any violation (not even the type captured by probability $\widehat{u}$). Then, when limiting auditing to tier 2, we rule out any collateral penalty from dependents whatsoever, thereby retaining the decreasing differences of the production profit augmented by $\widehat{u}$.

More generally, given state $\gamma=(g,U)\in\Gamma$, let $\widehat{S}_g$ be the set of suppliers in g that are *not* perfectly compliant (i.e., suppliers in $S_g\backslash\widehat{S}_g$ are perfectly compliant). Theorem G.1 remains valid, but (G.43) in Corollary G.1 now becomes

$$V^*(\gamma)=\pi(\gamma)-\widehat{u}wz|\widehat{S}_{g_\gamma}|-\widehat{c}_{\text{RP}}|U_\gamma|. \tag{G.44}$$

THEOREM G.2. *Under Condition 1 the following policy ξ^{**} is optimal at any state γ in which every tier-1 firm is vetted (perfectly compliant): for any nonterminal state $\gamma \neq \gamma_1$, let i be an LVUS in γ, then*

$$\xi^{**}(\gamma) = \begin{cases} \text{AD}(i), & \text{if } a + ud(\nabla(\gamma,i) - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \text{RP}, & \text{if } a + ud(\nabla(\gamma,i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases}. \tag{G.45}$$

We first identify an optimal policy in a general class of supply networks in Theorem G.3, the proof of which serves as the basis for the proof of Theorem G.4, an expanded version of Theorem G.2.

THEOREM G.3. *Let $\gamma_0 \in \Gamma$ be such that for any $\gamma \in R^+(\gamma_0)$ and $i \in U_\gamma$, any dependent of i in any state $\gamma' \in R^+(\gamma)$ is vetted (perfectly compliant). Under Assumptions D.1 and D.2, the following policy ξ^* is optimal in every state $\gamma \in R^+(\gamma_0)$:*

$$\xi^*(\gamma) = \begin{cases} \text{AD}(i), & \text{if } i \in U_\gamma, a + ud(\nabla(\gamma,i) - \widehat{u}wz) < \widehat{c}_{\text{RP}}, \text{ and } \nabla(\gamma,i) \leqslant \nabla(\gamma,i'), \forall i' \in U_\gamma \\ \text{RP}, & \text{if } a + ud(\nabla(\gamma,i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}, \forall i \in U_\gamma \end{cases}. \tag{G.46}$$

Proof. Let $\gamma \in R^+(\gamma_0)$. We prove the result by mathematical induction on the number of unaudited supplier in the state, $|U_\gamma|$. If $|U_\gamma| = 1$, let $i \in U_\gamma$, then $\widetilde{V}^*(\gamma, \text{AD}(i)) > \widetilde{V}^*(\gamma, \text{RP})$ iff

$$-a + (1-ud)V^*(\gamma \oplus i) + udV^*(\gamma \ominus i) > \pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - c_{\text{RP}} \tag{G.47}$$

iff

$$-a + (1-ud)(\pi(\gamma \oplus i) - \widehat{u}wz|\widehat{S}_{g_{\gamma\oplus i}}|) + ud(\pi(\gamma \ominus i) - \widehat{u}wz|\widehat{S}_{g_{\gamma\ominus i}}|) > \pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - c_{\text{RP}} \tag{G.48}$$

iff

$$-a + ud[\pi(\gamma \ominus i) - \widehat{u}wz(|\widehat{S}_{g_\gamma}| - 1)] > ud(\pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}|) - c_{\text{RP}} \tag{G.49}$$

iff

$$-a + ud(\nabla(\gamma,i) - \widehat{u}wz) < c_{\text{RP}} \tag{G.50}$$

which is the condition for action AD(i) under ξ^*. Therefore ξ^* is optimal at γ.

By way of mathematical induction, suppose ξ^* is optimal for all $\gamma' \in R^+(\gamma_0)$ such that $|U_{\gamma'}| \leqslant k \in \mathbb{N}^+$. Let $\gamma \in R^+(\gamma_0)$ be such that $|U_\gamma| = k+1$. We divide the proof of the induction step into two cases based on (G.46).

Case a $(a + ud(\nabla(\gamma,i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}, \forall i \in U_\gamma)$. We show that $\widetilde{V}^*(\gamma, \text{RP}) \geqslant \widetilde{V}^*(\gamma, \text{AD}(i))$ for any $i \in U_\gamma$, thus proving the optimality of the action RP when $u\nabla(\gamma,i) + a \geqslant c_{\text{RP}}, \forall i \in U_\gamma$ as Theorem G.3 prescribes. Let $i \in U_\gamma$. We first show two equalities: $V^*(\gamma \oplus i) = \widetilde{V}^*(\gamma \oplus i, \text{RP})$ and $V^*(\gamma \ominus i) = \widetilde{V}^*(\gamma \ominus i, \text{RP})$.
First consider the state $\gamma \oplus i$. Note $\pi(\gamma \oplus i) = \pi(\gamma)$ and for any $i' \in U_\gamma \backslash \{i\}$, $\pi(\gamma \oplus i \ominus i') = \pi(\gamma \ominus i')$. Then for any $i' \in U_{\gamma \oplus i} = U_\gamma \backslash \{i\}$,

$$\begin{aligned} a + ud(\nabla(\gamma \oplus i, i') - \widehat{u}wz) &= a + ud(\pi(\gamma \oplus i) - \pi(\gamma \oplus i \ominus i') - \widehat{u}wz) \\ &= a + ud(\pi(\gamma) - \pi(\gamma \ominus i') - \widehat{u}wz) = a + ud(\nabla(\gamma, i') - \widehat{u}wz) \geqslant c_{\text{RP}}. \end{aligned} \tag{G.51}$$

Hence by the definition of ξ^*, $\xi^*(\gamma \oplus i) = \text{RP}$, i.e., ξ^* prescribes the action RP in state $\gamma \oplus i$. But $|U_{\gamma \oplus i}| = k$. By invoking the induction hypothesis (that ξ^* is optimal at any state $\gamma' \in R^+(\gamma_0)$ in which $|U_{\gamma'}| \leqslant k$), we conclude that the action RP is optimal at state $\gamma \oplus i$. Therefore $V^*(\gamma \oplus i) = \widetilde{V}^*(\gamma \oplus i, \text{RP})$.

Next consider the state $\gamma \ominus i$. Since any dependent of i is perfectly compliant (and thus cannot be unaudited), $U_{\gamma\ominus i} = U_\gamma \setminus \{i\}$. By Assumption D.1, for any $i' \in U_{\gamma\ominus i}$,

$$a + ud(\nabla(\gamma \ominus i, i') - \widehat{u}wz) \geqslant a + ud(\nabla(\gamma, i') - \widehat{u}wz) \tag{G.52}$$

which is greater than or equal to $\widehat{c}_{\text{RP}}$ for any $i' \in U_\gamma$ by the premise of the current case. Hence $\xi^*(\gamma \ominus i) = \text{RP}$. But $|U_{\gamma\ominus i}| \leqslant k$. By invoking the induction hypothesis (that ξ^* is optimal at any state $\gamma' \in R^+(\gamma_0)$ in which $|U_{\gamma'}| \leqslant k$), we conclude that the action RP is optimal in state $\gamma \ominus i$. Therefore $V^*(\gamma \ominus i) = \widetilde{V}^*(\gamma \ominus i, \text{RP})$. Now

$$\begin{aligned}
\widetilde{V}^*(\gamma, \text{RP}) &= \pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - \widehat{c}_{\text{RP}}|U_\gamma| && \text{(G.53)}\\
&= \pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - \widehat{c}_{\text{RP}} - \widehat{c}_{\text{RP}}(|U_\gamma| - 1) && \text{(G.54)}\\
&\geqslant \pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - [a + ud(\pi(\gamma) - \pi(\gamma \ominus i) - \widehat{u}wz)] - \widehat{c}_{\text{RP}}(|U_\gamma| - 1) && \text{(G.55)}\\
&= -a + (1-ud)(\pi(\gamma \oplus i) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - \widehat{c}_{\text{RP}}|U_{\gamma\oplus i}|) + ud(\pi(\gamma \ominus i) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - \widehat{c}_{\text{RP}}|U_{\gamma\ominus i}|) && \text{(G.56)}\\
&= -a + (1-ud)\widetilde{V}^*(\gamma \oplus i, \text{RP}) + ud\widetilde{V}^*(\gamma \ominus i, \text{RP}) && \text{(G.57)}\\
&= -a + (1-ud)V^*(\gamma \oplus i) + udV^*(\gamma \ominus i) && \text{(G.58)}\\
&= \widetilde{V}^*(\gamma, \text{AD}(i)) && \text{(G.59)}
\end{aligned}$$

where (G.55) is by the assumption $a + ud(\pi(\gamma) - \pi(\gamma \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}$; (G.56) is by $|U_{\gamma\oplus i}| = |U_{\gamma\ominus i}| = |U_\gamma| - 1$ (any dependent of an unaudited supplier in γ is perfectly compliant so that $\gamma \ominus i$ has exactly one less unaudited supplier than γ); and (G.58) is by $V^*(\gamma \oplus i) = \widetilde{V}^*(\gamma \oplus i, \text{RP})$ and $V^*(\gamma \ominus i) = \widetilde{V}^*(\gamma \ominus i, \text{RP})$. Therefore the action RP is optimal when $u\nabla(\gamma, i) + a \geqslant c_{\text{RP}}, \forall i \in U_\gamma$.

Case b ($\exists i' \in U_\gamma$ such that $a + ud(\nabla(\gamma, i') - \widehat{u}wz) < \widehat{c}_{\text{RP}}$). Let $i \in U_\gamma$ be an LVUS in γ, i.e., $\nabla(\gamma, i) \leqslant \nabla(\gamma, j), \forall j \in U_\gamma$. We first show that $\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}^*(\gamma, \text{RP})$, then show that $\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}^*(\gamma, \text{AD}(i'))$ for any $i' \in U_\gamma$. With these we prove that if i is an LVUS in γ and $a + ud(\nabla(\gamma, i) - \widehat{u}wz) < \widehat{c}_{\text{RP}}$ then the optimal action to take in state γ is AD(i) as Theorem G.3 prescribes. Now

$$\begin{aligned}
\widetilde{V}^*(\gamma, \text{AD}(i)) &= -a + (1-ud)V^*(\gamma \oplus i) + udV^*(\gamma \ominus i) && \text{(G.60)}\\
&\geqslant -a + (1-ud)\widetilde{V}^*(\gamma \oplus i, \text{RP}) + ud\widetilde{V}^*(\gamma \ominus i, \text{RP}) && \text{(G.61)}\\
&= -a + (1-ud)(\pi(\gamma \oplus i) - \widehat{u}wz|\widehat{S}_{g_{\gamma\oplus i}}| - \widehat{c}_{\text{RP}}|U_{\gamma\oplus i}|) + ud(\pi(\gamma \ominus i) - \widehat{u}wz|\widehat{S}_{g_{\gamma\ominus i}}| - \widehat{c}_{\text{RP}}|U_{\gamma\ominus i}|) && \text{(G.62)}\\
&= -a + \pi(\gamma) - ud(\pi(\gamma) - \pi(\gamma \ominus i)) - (1-ud)\widehat{u}wz|\widehat{S}_{g_\gamma}| - ud\widehat{u}wz(|\widehat{S}_{g_\gamma}| - 1) - \widehat{c}_{\text{RP}}|U_\gamma| && \text{(G.63)}\\
&= \pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - \widehat{c}_{\text{RP}}|U_\gamma| - a - ud\nabla(\gamma, i) + ud\widehat{u}wz + \widehat{c}_{\text{RP}} && \text{(G.64)}\\
&= \widetilde{V}^*(\gamma, \text{RP}) - [a + ud(\nabla(\gamma, i) - \widehat{u}wz) - \widehat{c}_{\text{RP}}] && \text{(G.65)}\\
&> \widetilde{V}^*(\gamma, \text{RP}) && \text{(G.66)}
\end{aligned}$$

where (G.61) is by V^* being optimal; (G.63) is by $\pi(\gamma \oplus i) = \pi(\gamma)$; and (G.65) is by the premise of case b. We next show that $\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}^*(\gamma, \text{AD}(i'))$ for any $i' \in U_\gamma$. Let $i' \in U_\gamma$ such that i' is not symmetric with i. (If i' is symmetric with i, clearly $\widetilde{V}^*(\gamma, \text{AD}(i)) = \widetilde{V}^*(\gamma, \text{AD}(i'))$.) Since i is an LVUS in $\gamma \oplus i'$ and by the premise of case b $a + ud(\pi(\gamma \oplus i') - \pi(\gamma \oplus i' \ominus i) - \widehat{u}wz) = a + ud(\pi(\gamma) - \pi(\gamma \ominus i) - \widehat{u}wz) < \widehat{c}_{\text{RP}}$, by the induction hypothesis, $\xi^*(\gamma \oplus i') = \text{AD}(i)$. On the other hand, by Assumption D.2, i is an LVUS in $\gamma \ominus i'$; therefore[13]

$$\xi^*(\gamma \ominus i') = \begin{cases} \text{AD}(i), & \text{if } a + ud(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i) - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \text{AR}(i), & \text{if } a + ud(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \text{ and } a + udr < (u - \widehat{u})wz \\ \text{PP}, & \text{if } a + ud(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \text{ and } a + udr \geqslant (u - \widehat{u})wz \end{cases}. \tag{G.67}$$

We next look at the three cases in (G.67) separately. In each case we devise a policy $\widehat{\xi}$ so that the buyer's expected profit from first taking the action $\text{AD}(i)$ and following $\widehat{\xi}$ thereafter is at least as good as the expected profit from first taking $\text{AD}(i')$ and following the optimal policy ξ^* thereafter (ξ^* is optimal thereafter by the induction hypothesis). That is, $\widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi^*, \gamma, \text{AD}(i'))$. Since $\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)))$ and $\widetilde{V}(\xi^*, \gamma, \text{AD}(i')) = \widetilde{V}^*(\gamma, \text{AD}(i'))$, we must then have $\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}^*(\gamma, \text{AD}(i'))$ as desired. In each case we consider the following four events that together form a partition of the sample space:

$$H_{11} = \{\text{both } i \text{ and } i' \text{ would pass an audit}\} \tag{G.68}$$

$$H_{10} = \{i \text{ would pass an audit and } i' \text{ would not}\} \tag{G.69}$$

$$H_{01} = \{i \text{ would not pass an audit and } i' \text{ would}\} \tag{G.70}$$

$$H_{00} = \{\text{neither } i \text{ nor } i' \text{ would pass an audit}\}. \tag{G.71}$$

Case b(i) $(a + ud(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i) - \widehat{u}wz) < \widehat{c}_{\text{RP}})$. Let $\widehat{\xi} \in \Xi$ be the policy such that $\widehat{\xi}(\gamma \oplus i) = \widehat{\xi}(\gamma \ominus i) = \text{AD}(i')$ and $\widehat{\xi}(\gamma') = \xi^*(\gamma')$ for any $\gamma' \in \Gamma \backslash \{\gamma \oplus i, \gamma \ominus i\}$.

Conditional on H_{11}: The path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \oplus i \xrightarrow{\text{AD}(i')} \gamma \oplus i \oplus i' \tag{G.72}$$

while that by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \oplus i' \xrightarrow{\text{AD}(i)} \gamma \oplus i' \oplus i. \tag{G.73}$$

Note that $\gamma \oplus i \oplus i' = \gamma \oplus i' \oplus i$ and $\widehat{\xi}|_{R^+(\gamma \oplus i \oplus i')} = \xi^*|_{R^+(\gamma \oplus i' \oplus i)}$, so the expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{11}.

Conditional on H_{10}: The path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \oplus i \xrightarrow{\text{AD}(i')} \gamma \oplus i \ominus i' \tag{G.74}$$

while that by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{AD}(i)} \gamma \ominus i' \oplus i. \tag{G.75}$$

Note that $\gamma \oplus i \ominus i' = \gamma \ominus i' \oplus i$ and $\widehat{\xi}|_{R^+(\gamma \oplus i \ominus i')} = \xi^*|_{R^+(\gamma \ominus i' \oplus i)}$, so the expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{10}.

[13] If $u(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i)) + a \geqslant c_{\text{RP}}$ and $a + ur < uwz$, ξ^* prescribes auditing and rectify (if noncompliant) all unaudited suppliers in any sequence; here we choose i to audit next.

Conditional on H_{01} *or* H_{00}: Similarly we can show that the expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^*.

Therefore the unconditional expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ and that from first taking $\text{AD}(i')$ then following ξ^*, which are integrals of the respected conditional expected profits, must be equal; that is $\widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i)) = \widetilde{V}(\xi^*,\gamma,\text{AD}(i'))$. Therefore

$$\widetilde{V}^*(\gamma,\text{AD}(i)) \geqslant \widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i)) = \widetilde{V}(\xi^*,\gamma,\text{AD}(i')) = \widetilde{V}^*(\gamma,\text{AD}(i')) \tag{G.76}$$

where the induction hypothesis (that ξ^* is optimal at any state $\gamma' \in R^+(\gamma_0)$ with $|U_{\gamma'}| \leqslant k$) gives the last equality.

Case b(ii) $(a + ud(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}$ and $a + udr < (u - \widehat{u})wz)$. Let $\widehat{\xi} \in \Xi$ be the policy such that $\widehat{\xi}(\gamma \oplus i) = \text{AD}(i')$, $\widehat{\xi}(\gamma \ominus i) = \text{AR}(i')$, and $\widehat{\xi}(\gamma') = \xi^*(\gamma')$ for any $\gamma' \in \Gamma \backslash \{\gamma \oplus i, \gamma \ominus i\}$.

Conditional on H_{11}: Using the same steps as in case b(i) we can show the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{11}.

Conditional on H_{10}: The path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \oplus i \xrightarrow{\text{AD}(i')} \gamma \oplus i \ominus i' \tag{G.77}$$

while that by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{AR}(i)} \gamma \ominus i' \oplus i. \tag{G.78}$$

Note that $\gamma \oplus i \ominus i' = \gamma \ominus i' \oplus i$ and $\widehat{\xi}|_{R^+(\gamma \oplus i \ominus i')} = \xi^*|_{R^+(\gamma \ominus i' \oplus i)}$, so the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{10}.

Conditional on H_{01}: Similarly we can show that the expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^*.

Conditional on H_{00}: The path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \ominus i \xrightarrow{\text{AR}(i')} \gamma \ominus i \oplus i' \tag{G.79}$$

while that by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{AR}(i)} \gamma \ominus i' \oplus i. \tag{G.80}$$

Since $\pi(\gamma \ominus i' \oplus i) - \pi(\gamma \ominus i' \oplus i \ominus i'') = \pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i'') \geqslant \pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i)$ for any $i'' \in U_{\gamma \ominus i' \oplus i}$ (the last inequality is because i is an LVUS in $\gamma \ominus i'$, by Assumption D.2), and $a + ud(\pi(\gamma \ominus i') - \pi(\gamma \ominus i' \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}$ (premise of case b(ii)), we have $a + ud(\pi(\gamma \ominus i' \oplus i) - \pi(\gamma \ominus i' \oplus i \ominus i'') - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}$. Therefore $\xi^*(\gamma \ominus i' \oplus i) = \text{RP}$. Note that since $\widehat{\xi}|_{R^+(\gamma \ominus i \oplus i')} = \xi^*|_{R^+(\gamma \ominus i \oplus i')}$,

$$V(\widehat{\xi},\gamma \ominus i \oplus i') = V^*(\gamma \ominus i \oplus i') \geqslant \widetilde{V}^*(\gamma \ominus i \oplus i',\text{RP}). \tag{G.81}$$

On the other hand, since and dependent of i or i' is perfectly compliant and, in particular, $|U_{\gamma \ominus i \oplus i'}| = |U_{\gamma \ominus i' \oplus i}|$,

$$\widetilde{V}^*(\gamma \ominus i \oplus i',\text{RP}) - \widetilde{V}^*(\gamma \ominus i' \oplus i,\text{RP}) = \pi(\gamma \ominus i \oplus i') - \pi(\gamma \ominus i' \oplus i) = \pi(\gamma \ominus i) - \pi(\gamma \ominus i') \geqslant 0 \tag{G.82}$$

Together they imply

$$V(\widehat{\xi},\gamma\ominus i\oplus i')\geqslant \widetilde{V}^*(\gamma\ominus i'\oplus i,\text{RP})=V^*(\gamma\ominus i'\oplus i) \tag{G.83}$$

where the last equality is because $\xi^*(\gamma\ominus i'\oplus i)=\text{RP}$. Therefore the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is greater than or equal to that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{10}.

Therefore the unconditional expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is greater than or equal to that from first taking $\text{AD}(i')$ then following ξ^*; that is $\widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i))\geqslant\widetilde{V}(\xi^*,\gamma,\text{AD}(i'))$. Therefore

$$\widetilde{V}^*(\gamma,\text{AD}(i))\geqslant\widetilde{V}(\widehat{\xi},\gamma,\text{AD}(i))\geqslant\widetilde{V}(\xi^*,\gamma,\text{AD}(i'))=\widetilde{V}^*(\gamma,\text{AD}(i')) \tag{G.84}$$

where the induction hypothesis gives the last equality.

Case b(iii) $(a+ud(\pi(\gamma\ominus i')-\pi(\gamma\ominus i'\ominus i)-\widehat{u}wz)\geqslant\widehat{c}_{\text{RP}}$ and $a+udr\geqslant(\widehat{u}-u)wz)$. Let $\widehat{\xi}\in\Xi$ be the policy such that (1) $\widehat{\xi}(\gamma\oplus i)=\text{AD}(i')$, (2) for any $\gamma'\in R^+(\gamma\ominus i)$ such that $i'\in U_{\gamma'}$, $\widehat{\xi}(\gamma')=\xi^*(\gamma'\ominus i')$, and (3) $\widehat{\xi}(\gamma')=\xi^*(\gamma')$ for any other state γ' (i.e., $\gamma'\in\Gamma\backslash\{\gamma\oplus i\}\backslash\{\gamma''\in R^+(\gamma\ominus i):i'\in U_{\gamma''}\}$).

Conditional on H_{11}: Using the same corresponding steps as in case b(i) we can show the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is the same as that from first taking $\text{AD}(i')$ then following ξ^* conditional on H_{11}.

Conditional on H_{10}: Since i is an LVUS in $\gamma\ominus i'$, by the premise of case b(iii), any unaudited supplier i'' in state $\gamma\oplus i\ominus i'$ must have $a+ud(\nabla(\gamma\oplus i\ominus i',i'')-\widehat{u}wz)=a+ud(\nabla(\gamma\ominus i',i'')-\widehat{u}wz)\geqslant\widehat{c}_{\text{RP}}$, then the induction hypothesis implies $\xi^*(\gamma\oplus i\ominus i')=\text{PP}$. By the definition of $\widehat{\xi}$, $\widehat{\xi}(\gamma\oplus i\ominus i')=\xi^*(\gamma\oplus i\ominus i')$. Therefore $\widehat{\xi}(\gamma\oplus i\ominus i')=\text{PP}$. Then the path of the state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma\xrightarrow{\text{AD}(i)}\gamma\oplus i\xrightarrow{\text{AD}(i')}\gamma\oplus i\ominus i'\xrightarrow{\text{PP}}. \tag{G.85}$$

The path of the state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma\xrightarrow{\text{AD}(i')}\gamma\ominus i'\xrightarrow{\text{PP}}. \tag{G.86}$$

Note that $\pi(\gamma\oplus i\ominus i')=\pi(\gamma\ominus i')$, so the only difference in the conditional expected profit between the above two paths is the additional cost a of carrying out one more audit in (G.85) (since i is compliant on H_{10} it will not incur any penalty from violation later on).

Conditional on H_{01}: The path of the state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma\xrightarrow{\text{AD}(i)}\gamma\ominus i \tag{G.87}$$

while the path of the state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma\xrightarrow{\text{AD}(i')}\gamma\oplus i'\xrightarrow{\text{AD}(i)}\gamma\oplus i'\ominus i. \tag{G.88}$$

Note that the definition of $\widehat{\xi}$ means that the path subsequent to $\gamma\ominus i$ in (G.87) and that subsequent to $\gamma\oplus i'\ominus i$ in (G.88) will be identical except that i' will remain unaudited in all subsequent states in (G.87) while it is vetted in (G.88). Since on H_{01} i' is compliant the only difference in the conditional expected profit between the above two paths is the additional cost a of carrying out one more audit in (G.88) (since i' is compliant on H_{10}, even if unaudited, it will not incur any penalty from violation later on).

Conditional on H_{00}: The path of the state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \ominus i \xrightarrow{\text{PP}} \tag{G.89}$$

where $\widehat{\xi}(\gamma \ominus i) = \xi^*(\gamma \ominus i \ominus i') = \text{PP}$ by the premise of the current case and Assumption D.1 (so that $a + ud(\pi(\gamma \ominus i \ominus i') - \pi(\gamma \ominus i \ominus i' \ominus i'') - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}, \forall i'' \in U_{\gamma \ominus i \ominus i'}$), while the path of the state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^* is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{PP}} . \tag{G.90}$$

Therefore conditional on H_{00} the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is greater than that from first taking $\text{AD}(i')$ then following ξ^* by precisely $\pi(\gamma \ominus i) - \pi(\gamma \ominus i') \geqslant 0$.

Therefore the unconditional expected profit at γ from first taking $\text{AD}(i)$ then following policy $\widehat{\xi}$ is greater than or equal to that from first taking $\text{AD}(i')$ then following ξ^*; that is $\widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi^*, \gamma, \text{AD}(i'))$. Therefore

$$\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}(\widehat{\xi}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi^*, \gamma, \text{AD}(i')) = \widetilde{V}^*(\gamma, \text{AD}(i')) \tag{G.91}$$

where the induction hypothesis yields the last equality.

To sum up, in all cases b(i)–b(iii), $\widetilde{V}^*(\gamma, \text{AD}(i)) \geqslant \widetilde{V}^*(\gamma, \text{AD}(i'))$.

ξ^* is optimal at γ. $\square$

Theorem G.2 is a shortened version of Theorem G.4.

THEOREM G.4. *Under Condition 1 the following policy ξ^{**} is optimal at any state γ in which every tier-1 firm is vetted: for any nonterminal state $\gamma \neq \gamma_1$, let i be an LVUS in γ, then*

$$\xi^{**}(\gamma) = \begin{cases} \text{AD}(i), & \textit{if } a + ud(\nabla(\gamma, i) - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \text{RP}, & \textit{if } a + ud(\nabla(\gamma, i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases} \tag{G.92}$$

and for γ_1 and $i \in U_{\gamma_1}$,

$$\xi^{**}(\gamma_1) = \begin{cases} \text{AD}(i), & \textit{if } \dfrac{1}{1+ud}[a + ud(\nabla(\gamma, i) - \widehat{u}wz)] + \dfrac{ud}{1+ud}[a + ud(\pi(\gamma \ominus i) - \widehat{u}wz)] < \widehat{c}_{\text{RP}} \\ \text{RP}, & \textit{if } \dfrac{1}{1+ud}[a + ud(\nabla(\gamma, i) - \widehat{u}wz)] + \dfrac{ud}{1+ud}[a + ud(\pi(\gamma \ominus i) - \widehat{u}wz)] \geqslant \widehat{c}_{\text{RP}}. \end{cases} . \tag{G.93}$$

The policy ξ^{**} differs from ξ^* only at state γ_1 where Assumption D.1 fails. At state γ_1, ξ^{**} prescribes $\text{AD}(i)$ in a larger region of the parameter space than ξ^* does, since ξ^{**} takes into account the fact that if the buyer drops i, the last remaining unaudited supplier will be even less valuable. The buyer has less incentive to keep the supply network operating in state γ_1 than in states in which decreasing differences hold.

Proof of Theorem G.4. Since we limit to states in which all tier-1 firms are perfectly compliant, (1) by Proposition 2, Assumption D.2 holds, and (2) any dependent of an unaudited supplier is perfectly compliant. Under Condition 1, among all states we consider here the only state at which Assumption D.1 fails is γ_1, the induction proof of Theorem G.3 applies directly by replacing ξ^* with ξ^{**}, with two exceptions: (1) at γ_1 itself, at which state we show the optimality of ξ^{**} separately, and (2) at state $\gamma_2 = (g, U)$ where $g = (\{\text{A}, \text{B}\}, \{1\}, \{3\}, \{2\})$ and $U = \{1, 2, 3\}$, and if the LVUS, 1, in γ_2 satisfies $a + ud(\nabla(\gamma_2, 1) - \widehat{u}wz) < \widehat{c}_{\text{RP}}$. We will go on to show why the induction proof still applies in the second case.

ξ^{**} is optimal at γ_1. Set $\gamma = \gamma_1$. Call the two symmetric tier-2 suppliers in γ_1 i and i'. To analyze the decision at γ we first consider the profits in state $\gamma \oplus i$ and $\gamma \ominus i$. In state $\gamma \oplus i$ the only unaudited supplier is i'. The decision is between $\text{AD}(i')$ (with expected profit $-a + (1-ud)(\pi(\gamma \oplus i \oplus i') - 2\widehat{u}wz) + ud(\pi(\gamma \oplus i \ominus i') - \widehat{u}wz))$ and RP (with expected profit $\pi(\gamma \oplus i \oplus i') - 2\widehat{u}wz - \widehat{c}_{\text{RP}}$). Therefore

$$V^*(\gamma \oplus i) = \begin{cases} \begin{array}{r} -a + (1-ud)(\pi(\gamma \oplus i \oplus i') - 2\widehat{u}wz) \\ +ud(\pi(\gamma \oplus i \ominus i') - \widehat{u}wz) \end{array}, & \text{if } a + ud(\nabla(\gamma \oplus i, i') - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \pi(\gamma \oplus i) - 2\widehat{u}wz - \widehat{c}_{\text{RP}}, & \text{if } a + ud(\nabla(\gamma \oplus i, i') - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases} \tag{G.94}$$

$$= \begin{cases} \begin{array}{r} -a + (1-ud)(\pi(\gamma) - 2\widehat{u}wz) \\ +ud(\pi(\gamma \ominus i') - \widehat{u}wz) \end{array}, & \text{if } a + ud(\nabla(\gamma \oplus i, i') - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \pi(\gamma) - 2\widehat{u}wz - \widehat{c}_{\text{RP}}, & \text{if } a + ud(\nabla(\gamma \oplus i, i') - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases}. \tag{G.95}$$

Similary, in state $\gamma \ominus i$ the only unaudited supplier is i'. The decision is between $\text{AD}(i')$ (with expected profit $-a + (1-ud)(\pi(\gamma \ominus i \oplus i') - \widehat{u}wz))$ and RP (with expected profit $\pi(\gamma \ominus i \oplus i') - \widehat{u}wz - \widehat{c}_{\text{RP}}$). Therefore

$$V^*(\gamma \ominus i) = \begin{cases} -a + (1-ud)(\pi(\gamma \ominus i \oplus i') - \widehat{u}wz), & \text{if } a + ud(\pi(\gamma \ominus i \oplus i') - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \pi(\gamma \ominus i \oplus i') - \widehat{u}wz - \widehat{c}_{\text{RP}}, & \text{if } a + ud(\pi(\gamma \ominus i \oplus i') - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases} \tag{G.96}$$

$$= \begin{cases} -a + (1-ud)(\pi(\gamma \ominus i) - \widehat{u}wz), & \text{if } a + ud(\pi(\gamma \ominus i \oplus i') - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \pi(\gamma \ominus i) - \widehat{u}wz - \widehat{c}_{\text{RP}}, & \text{if } a + ud(\pi(\gamma \ominus i \oplus i') - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases}. \tag{G.97}$$

By Proposition C.5 we algebraically verify that

$$\pi(\gamma) - \pi(\gamma \ominus i') > \pi(\gamma \ominus i') = \pi(\gamma \ominus i) \tag{G.98}$$

(which is how Assumption D.1 is violated). By (G.95) and (G.97) we obtain

$$\widetilde{V}^*(\gamma, \text{AD}(i)) = -a + (1-ud)V^*(\gamma \oplus i) + udV^*(\gamma \ominus i) \tag{G.99}$$

$$= \begin{cases} \begin{array}{r} -2a + (1-ud)[(1-ud)(\pi(\gamma) - 2\widehat{u}wz) \\ +ud(\pi(\gamma \ominus i') - \widehat{u}wz)] \\ +ud(1-ud)(\pi(\gamma \ominus i) - \widehat{u}wz), \end{array} & \text{if } a + ud(\nabla(\gamma, i') - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \begin{array}{r} -a + (1-ud)(\pi(\gamma) - 2\widehat{u}wz - \widehat{c}_{\text{RP}}) \\ +ud[-a + (1-ud)(\pi(\gamma \ominus i) - \widehat{u}wz)], \end{array} & \text{if } \begin{array}{l} a + ud(\pi(\gamma \ominus i) - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \leqslant a + ud(\delta(\gamma, i') - \widehat{u}wz) \end{array} \\ \begin{array}{r} -a + (1-ud)(\pi(\gamma) - 2\widehat{u}wz - \widehat{c}_{\text{RP}}) \\ +ud(\pi(\gamma \ominus i) - \widehat{u}wz - \widehat{c}_{\text{RP}}), \end{array} & \text{if } a + ud(\pi(\gamma \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases} \tag{G.100}$$

$$= \begin{cases} \begin{array}{l} -2a + (1-ud)^2(\pi(\gamma) - 2\widehat{u}wz) \\ +2ud(1-ud)(\pi(\gamma \ominus i) - \widehat{u}wz), \end{array} & \text{if } a + ud(\nabla(\gamma, i') - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \begin{array}{r} -a + (1-ud)(\pi(\gamma) - 2\widehat{u}wz - \widehat{c}_{\text{RP}}) \\ +ud[-a + (1-ud)(\pi(\gamma \ominus i) - \widehat{u}wz)], \end{array} & \text{if } \begin{array}{l} a + ud(\pi(\gamma \ominus i) - \widehat{u}wz) < \widehat{c}_{\text{RP}} \\ \leqslant a + ud(\delta(\gamma, i') - \widehat{u}wz) \end{array} \\ -a - \widehat{c}_{\text{RP}} + (1-ud)(\pi(\gamma) - 2\widehat{u}wz) + ud(\pi(\gamma \ominus i) - \widehat{u}wz), & \text{if } a + ud(\pi(\gamma \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}} \end{cases}. \tag{G.101}$$

On the other hand $\widetilde{V}^*(\gamma, \text{RP}) = \pi(\gamma) - \widehat{u}wz|\widehat{S}_{g_\gamma}| - \widehat{c}_{\text{RP}}|U_\gamma| = \pi(\gamma) - 2\widehat{u}wz - 2\widehat{c}_{\text{RP}}$. Hence $\widetilde{V}^*(\gamma, \text{AD}(i)) > \widetilde{V}^*(\gamma, \text{RP})$ if and only if one of the following three (mutually exclusive) conditions holds:

(*a*) $a + ud(\nabla(\gamma, i') - \widehat{u}wz) < \widehat{c}_{\text{RP}}$ and $-2a + (1-ud)^2(\pi(\gamma) - 2\widehat{u}wz) + 2ud(1-ud)(\pi(\gamma \ominus i') - \widehat{u}wz) > \pi(\gamma) - 2\widehat{u}wz - 2\widehat{c}_{\text{RP}}$;

(*b*) $a + ud(\pi(\gamma \ominus i) - \widehat{u}wz) < \widehat{c}_{\text{RP}} \leqslant a + ud(\nabla(\gamma, i') - \widehat{u}wz)$ and $-a + (1-ud)(\pi(\gamma) - 2\widehat{u}wz - \widehat{c}_{\text{RP}} + ud[-a + (1-ud)(\pi(\gamma \ominus i) - \widehat{u}wz)] > \pi(\gamma) - 2\widehat{u}wz - 2\widehat{c}_{\text{RP}}$;

(*c*) $a + ud(\pi(\gamma \ominus i) - \widehat{u}wz) \geqslant \widehat{c}_{\text{RP}}$ and $-a - \widehat{c}_{\text{RP}} + (1-ud)(\pi(\gamma) - 2\widehat{u}wz) + ud(\pi(\gamma \ominus i) - \widehat{u}wz) > \pi(\gamma) - 2\widehat{u}wz - 2\widehat{c}_{\text{RP}}$.

In (a), the second inequality is equivalent to

$$2[a+ud(\nabla(\gamma,i')-2\widehat{u}wz)]-(ud)^2[(\pi(\gamma)-\pi(\gamma\ominus i'))-\pi(\gamma\ominus i')]<2\widehat{c}_{\text{RP}} \tag{G.102}$$

which is implied by the first inequality in (a) and (G.98). So (a) can be simplified to just the first inequality $a+ud(\nabla(\gamma,i')-\widehat{u}wz)<\widehat{c}_{\text{RP}}$. In (b), the last inequality is equivalent to

$$[a+ud(\nabla(\gamma,i)-\widehat{u}wz)]+ud(a+ud(\pi(\gamma\ominus i)-\widehat{u}wz))<(1+ud)\widehat{c}_{\text{RP}} \tag{G.103}$$

or

$$\frac{1}{1+ud}[a+ud(\nabla(\gamma,i)-\widehat{u}wz)]+\frac{ud}{1+ud}[a+ud(\pi(\gamma\ominus i)-\widehat{u}wz)]<\widehat{c}_{\text{RP}}. \tag{G.104}$$

Note that (G.104) and the second inequality $\widehat{c}_{\text{RP}}\leqslant a+ud(\nabla(\gamma,i')-\widehat{u}wz)$ implies the first inequality $a+ud(\pi(\gamma\ominus i)-\widehat{u}wz)<\widehat{c}_{\text{RP}}$. So (b) can be simplified to

$$\frac{1}{1+ud}[a+ud(\nabla(\gamma,i)-\widehat{u}wz)]+\frac{ud}{1+ud}[a+ud(\pi(\gamma\ominus i)-\widehat{u}wz)]<\widehat{c}_{\text{RP}}\leqslant a+ud(\nabla(\gamma,i')-\widehat{u}wz). \tag{G.105}$$

In (c), the second inequality is equivalent to

$$a+ud(\nabla(\gamma,i)-\widehat{u}wz)<\widehat{c}_{\text{RP}} \tag{G.106}$$

directly contradicting the first inequality; (c) can never hold. Therefore that one of the above three conditions holds is equivalent to that one of the following two conditions holds:

(a) $a+ud(\nabla(\gamma,i')-\widehat{u}wz)<\widehat{c}_{\text{RP}}$;

(b) $\frac{1}{1+ud}[a+ud(\nabla(\gamma,i)-\widehat{u}wz)]+\frac{ud}{1+ud}[a+ud(\pi(\gamma\ominus i)-\widehat{u}wz)]<\widehat{c}_{\text{RP}}\leqslant a+ud(\nabla(\gamma,i')-\widehat{u}wz)$

which by (G.98) is equivalent to just

$$\frac{1}{1+ud}[a+ud(\nabla(\gamma,i)-\widehat{u}wz)]+\frac{ud}{1+ud}[a+ud(\pi(\gamma\ominus i)-\widehat{u}wz)]<\widehat{c}_{\text{RP}}. \tag{G.107}$$

This shows the optimality of ξ^{**} at γ_1 as in (G.93).

ξ^{**} is optimal at γ_2 The only case to show is when the LVUS i of γ_2 satisfies $a+ud(\nabla(\gamma_2,i)-\widehat{u}wz)<\widehat{c}_{\text{RP}}$. The proof is analogous to case b in the proof of Theorem G.3 by replacing ξ^* with ξ^{**}; here we only point out the differences:

- Since now $\gamma=\gamma_2$ and i' is the shared supplier in γ_2, $\gamma\ominus i'=\gamma_1$. Hence by the induction hypothesis

$$\xi^{**}(\gamma\ominus i')=\begin{cases}\text{AD}(i), & \text{if } F<c_{\text{RP}};\\ \text{AR}(i), & \text{if } F\geqslant c_{\text{RP}} \text{ and } a+udr<(u-\widehat{u})wz;\\ \text{PP}, & \text{if } F\geqslant c_{\text{RP}} \text{ and } a+udr\geqslant(u-\widehat{u})wz;\end{cases} \tag{G.108}$$

where $F=\frac{1}{1+ud}[a+ud(\nabla(\gamma\ominus i',i)-\widehat{u}wz)]+\frac{ud}{1+ud}[a+ud(\pi(\gamma\ominus i'\ominus i)-\widehat{u}wz)]$. We redefine the three subcases b(i), b(ii), and b(iii) in the proof by the three cases for $\xi^{**}(\gamma\ominus i')$ in (G.108) (i.e., replace $a+ud(\nabla(\gamma\ominus i',i)-\widehat{u}wz)$ in the original condition for each subcase by F).

- In subcases b(ii) and b(iii), owing to (G.98),

$$\nabla(\gamma_1,i')>\nabla(\gamma_1\ominus i,i'). \tag{G.109}$$

$F\geqslant c_{\text{RP}}$ implies $a+ud(\nabla(\gamma\ominus i',i)-\widehat{u}wz)\geqslant\widehat{c}_{\text{RP}}$.

- In subcase b(iii) ($F \geqslant \widehat{c}_{\text{RP}}$ and $a + udr \geqslant (u - \widehat{u})wz$) conditional on H_{00} the path of state transition by taking $\text{AD}(i)$ at state γ then following policy $\widehat{\xi}$ is

$$\gamma \xrightarrow{\text{AD}(i)} \gamma \ominus i \tag{G.110}$$

while the path of state transition by taking $\text{AD}(i')$ at state γ then following policy ξ^{**} is

$$\gamma \xrightarrow{\text{AD}(i')} \gamma \ominus i' \xrightarrow{\text{PP}} . \tag{G.111}$$

Here by the definition of $\widehat{\xi}$ and the induction hypothesis one of two actions could be taken subsequent to (G.110) ($\pi(\gamma \ominus i \ominus i') = \frac{1}{64}\frac{(\alpha - v_T)^2}{\beta}$ by Proposition C.5):

A. If $a + ud(\pi(\gamma \ominus i \ominus i') - \widehat{u}wz) = a + ud\left(\frac{1}{64}\frac{(\alpha - v_T)^2}{\beta} - \widehat{u}wz\right) \geqslant \widehat{c}_{\text{RP}}$, then $\widehat{\xi}(\gamma \ominus i) = \xi^{**}(\gamma \ominus i \ominus i') = \text{PP}$;

B. If $a + ud(\pi(\gamma \ominus i \ominus i') - \widehat{u}wz) = a + ud\left(\frac{1}{64}\frac{(\alpha - v_T)^2}{\beta} - \widehat{u}wz\right) < \widehat{c}_{\text{RP}}$, then $\widehat{\xi}(\gamma \ominus i) = \xi^{**}(\gamma \ominus i \ominus i') = \text{AD}(i'')$ where i'' is the only supplier in $U_{\gamma \ominus i \ominus i'}$.

In sub-subcase (b)(iii)A the original proof applies. In sub-subcase (b)(iii)B following (G.110) the action is $\text{AD}(i'')$ with two possible consequences: that i'' passes the audit leading to $\gamma \ominus i \oplus i''$ and that i'' fails the audit leading to $\gamma \ominus i \ominus i''$. Note that in either case the definition of $\widehat{\xi}$ prescribes PP afterward. Therefore, the expected profit subsequent to $\gamma \ominus i$ in (G.110) is

$$\begin{aligned} &-a + (1 - ud)(\pi(\gamma \ominus i \oplus i'') - \widehat{u}wz) + ud\pi(\gamma \ominus i \ominus i'') - wz \\ &\qquad = -a + \left[(1 - ud)\frac{25}{576} + ud\frac{1}{36}\right]\frac{(\alpha - v_T)^2}{\beta} - (1 - ud)\widehat{u}wz - wz \end{aligned} \tag{G.112}$$

where the $-wz$ comes from that in event H_{00} we know i' is noncompliant and the equality results from substituting the values of the production profits according to Proposition C.5. On the other hand the expected profit subsequent to $\gamma \ominus i'$ in (G.111) is

$$\pi(\gamma \ominus i') - uwz - wz = \frac{1}{25}\frac{(\alpha - v_T)^2}{\beta} - uwz - wz \tag{G.113}$$

where the $-uwz$ is due to i'' remaining unaudited, the $-wz$ is due to i being noncompliant, and the equality results from substituting the value of the production profit according to Proposition C.5. The difference between (G.112) and (G.113) is equal to

$$(u - \widehat{u})wz - \left[a + ud\left(\frac{1}{64}\frac{(\alpha - v_T)^2}{\beta} - \widehat{u}wz\right)\right] + \frac{49}{14,400}\frac{(\alpha - v_T)^2}{\beta}. \tag{G.114}$$

But the premise of sub-subcase (b)(iii)B is that $a + ud\left(\frac{1}{64}\frac{(\alpha - v_T)^2}{\beta} - \widehat{u}wz\right) < \widehat{c}_{\text{RP}}$ and a premise of subcase b(iii) is $a + udr \geqslant (u - \widehat{u})wz$, implying $\widehat{c}_{\text{RP}} = (\widehat{u} - u)wz$, so (G.114) is positive.

Therefore the expected profit at γ from first taking $\text{AD}(i)$ then following $\widehat{\xi}$ is greater than or equal to that from first taking $\text{AD}(i')$ then following ξ^{**} conditional on H_{00}. This completes the proof. □

Corollary 2 holds exactly as in the base model.

G.3. Supplier Choice When Auditing One Firm

Let $\gamma \in \Gamma$ and $i \in U_\gamma$. We revise the definitions of the two thresholds $z_p(\gamma,i)$ and $z_r(\gamma,i)$ for z as

$$z_p(\gamma,i) = \frac{a+ud\nabla(\gamma,i)}{w\{ud[\widehat{u}(|S_{g_\gamma}|-|S_{g_{\gamma\ominus i}}|)+(u-\widehat{u})(|U_\gamma|-|U_{\gamma\ominus i}|-1)]+(u-\widehat{u})]\}} \tag{G.115}$$

$$z_r(\gamma,i) = \frac{\nabla(\gamma,i)-r}{w[(u-\widehat{u})(|U_\gamma|-|U_{\gamma\ominus i}|-1)+\widehat{u}(|S_{g_\gamma}|-|S_{g_{\gamma\ominus i}}|)]}. \tag{G.116}$$

Proposition G.3. *At nonterminal state γ, suppose the buyer can at most audit (* AD *or* AR*) one supplier, then* PP*. The optimal decision is*

(*a*) PP *if and only if $z \leqslant \frac{a+udr}{w(u-\widehat{u})}$ and $z \leqslant z_p(\gamma,i')$ for every $i' \in U_\gamma$;*

(*b*) AR(i) *(for any $i \in U_\gamma$) if and only if $z > \frac{a+udr}{w(u-\widehat{u})}$, and $z \leqslant z_r(\gamma,i')$ for every $i' \in U_\gamma$.*

(*c*) AD(i) *if and only if $z > z_p(\gamma,i)$, $z > z_r(\gamma,i)$ and i solves*

$$\max_{i\in U_\gamma}\left\{\pi(\gamma\ominus i) - wz\left[\widehat{u}|S_{g_{\gamma\ominus i}}| + (u-\widehat{u})|U_{\gamma\ominus i}|\right]\right\}. \tag{G.117}$$

Proof. Let ξ_{PP} be the policy that maps any state in Γ to the action PP. Then for $i \in U_\gamma$,

$$\begin{aligned}
\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) &= -a + (1-ud)V(\xi_{\text{PP}},\gamma\oplus i) + udV(\xi_{\text{PP}},\gamma\ominus i) &\text{(G.118)}\\
&= -a + (1-ud)(\pi(\gamma\oplus i) - wz[\widehat{u}|S_{g_{\gamma\oplus i}}| + (u-\widehat{u})|U_{\gamma\oplus i}|)\\
&\quad + ud\{\pi(\gamma\ominus i) - wz[\widehat{u}|S_{\gamma_{\gamma\ominus i}}| + (u-\widehat{u})|U_{\gamma\ominus i}|)]\} &\text{(G.119)}\\
&= -a + (1-ud)(\pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| + (u-\widehat{u})|U_\gamma|-1)\\
&\quad + ud\{\pi(\gamma\ominus i) - wz[\widehat{u}|S_{g_{\gamma\ominus i}}| + (u-\widehat{u})|U_{\gamma\ominus i}|)]\} &\text{(G.120)}
\end{aligned}$$

$$\begin{aligned}
\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i)) &= -a - udr + \pi(\gamma\oplus i) - wz(\widehat{u}|S_{g_{\gamma\oplus i}}| + (u-\widehat{u})|U_{\gamma\oplus i}|) &\text{(G.121)}\\
&= -a - udr + \pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| + (u-\widehat{u})(|U_\gamma|-1)] &\text{(G.122)}
\end{aligned}$$

and

$$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP}) = \pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| + (u-\widehat{u})|U_\gamma|]. \tag{G.123}$$

Note that $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i))$ is independent of i.

Therefore $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i'))$ for any $i' \in U_\gamma$ iff

$$\begin{aligned}
&-a + (1-ud)\{\pi(\gamma) - wz[\widehat{u}|S_{g_{\gamma\ominus i}}| + (u-\widehat{u})(|U_\gamma|-1)]\} + ud\{\pi(\gamma\ominus i) - wz[\widehat{u}|S_{g_{\gamma\ominus i}}| + (u-\widehat{u})|U_{\gamma\ominus i}|]\}\\
&\qquad > -a - udr + \pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| + (u-\widehat{u}(|U_\gamma|-1)] \qquad \text{(G.124)}
\end{aligned}$$

which is equivalent to

$$wz[\widehat{u}(|S_{g_\gamma}| - |S_{g_{\gamma\ominus i}}| + (u-\widehat{u})(|U_\gamma| - |U_{\gamma\ominus i}|-1)] > \nabla(\gamma,i) - r \tag{G.125}$$

or $z > z_r(\gamma,i)$.

$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i)) > \widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$ iff

$$\begin{aligned}
&-a + (1-ud)(\pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| + (u-\widehat{u})|U_\gamma|-1) + ud\{\pi(\gamma\ominus i) - wz[\widehat{u}|S_{g_{\gamma\ominus i}}| + (u-\widehat{u})|U_{\gamma\ominus i}|)]\}\\
&\qquad > \pi(\gamma) - wz[\widehat{u}|S_{g_\gamma}| + (u-\widehat{u})|U_\gamma|] \qquad \text{(G.126)}
\end{aligned}$$

which is equivalent to

$$uw\{ud[\widehat{u}(|S_{g_\gamma}|-|S_{g_{\gamma\ominus i}}|)+(u-\widehat{u})(|U_\gamma|-|U_{\gamma\ominus i}|)]+(1-ud)(u-\widehat{u})\}>a+ud\nabla(\gamma,i) \tag{G.127}$$

or $z>z_p(\gamma,i)$.

$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i))\geqslant\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i'))$ for $i'\in U_\gamma$ iff

$$\begin{aligned}&-a+(1-ud)(\pi(\gamma)-wz[\widehat{u}|S_{g_\gamma}|+(u-\widehat{u})|U_\gamma|-1)+ud\{\pi(\gamma\ominus i)-wz[\widehat{u}|S_{g_{\gamma\ominus i}}|+(u-\widehat{u})|U_{\gamma\ominus i}|)]\}\\&\quad\geqslant -a+(1-ud)(\pi(\gamma)-wz[\widehat{u}|S_{g_\gamma}|+(u-\widehat{u})|U_\gamma|-1)+ud\{\pi(\gamma\ominus i')-wz[\widehat{u}|S_{g_{\gamma\ominus i'}}|+(u-\widehat{u})|U_{\gamma\ominus i'}|)]\}\end{aligned} \tag{G.128}$$

which is equivalent to

$$\pi(\gamma\ominus i)-wz[\widehat{u}|S_{g_{\gamma\ominus i}}|+(u-\widehat{u})|U_{\gamma\ominus i}|)]\geqslant\pi(\gamma\ominus i')-wz[\widehat{u}|S_{g_{\gamma\ominus i'}}|+(u-\widehat{u})|U_{\gamma\ominus i'}|)]. \tag{G.129}$$

$\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i))>\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$ iff

$$-a-udr+\pi(\gamma)-wz[\widehat{u}|S_{g_\gamma}|+(u-\widehat{u})(|U_\gamma|-1)]>\pi(\gamma)-wz[\widehat{u}|S_{g_\gamma}|+(u-\widehat{u})|U_\gamma|] \tag{G.130}$$

which is equivalent to $z>\frac{a+udr}{w(u-\widehat{u})}$.

The optimal decision is PP iff $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})\geqslant\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i))$ and $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})\geqslant\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i))$ for any $i\in U_\gamma$. This gives part (*a*). The optimal decision is AD(i) iff $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i))>\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i'))$, $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i))>\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$, and $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i))\geqslant\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i'))$ for any $i'\in U_\gamma$. This gives part (*c*). The optimal decision is AR(i) iff $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i))\geqslant\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AD}(i'))$ for any $i'\in U_\gamma$ and $\widetilde{V}(\xi_{\text{PP}},\gamma,\text{AR}(i))>\widetilde{V}(\xi_{\text{PP}},\gamma,\text{PP})$. This gives part (*b*). □

PROPOSITION G.4. *At nonterminal state γ, suppose the buyer can audit (*AD *or* AR*) at most one supplier, before proceeding to production (*PP*). There exist two (possibly coinciding) thresholds $\underline{z}\leqslant\bar{z}$ for penalty z such that*

(*a*) *If $z\leqslant\underline{z}$ the optimal decision is* PP*;*

(*b*) *If $\underline{z}<z\leqslant\bar{z}$ the optimal decision is* AR(i) *for any $i\in U_\gamma$;*

(*c*) *If $z>\bar{z}$ the optimal decision is* AD(i) *where i solves* (G.117).

Proof. By Proposition G.3 the optimal decision depends on the ordering of three thresholds for z:

$$\frac{a+udr}{w(u-\widehat{u})},\quad \underline{z}_r(\gamma)\equiv\min_{i\in U_\gamma}z_r(\gamma,i),\quad \underline{z}_p(\gamma)\equiv\min_{i\in U_\gamma}z_p(\gamma,i). \tag{G.131}$$

In the following we enumerate all but one possible orderings of the three thresholds and verify that they are consistent with the property we describe in Proposition G.4. We then show the remaining one ordering can never arise. In the following the supplier i in AR(i) can be any $i\in U_\gamma$ and the supplier i in AD(i) is given by (G.117). (The identity of supplier i may change as z changes.) We consider two cases as follows:

(*a*) $\frac{a+udr}{w(u-\widehat{u})}\geqslant\underline{z}_r(\gamma)$. Then by Proposition G.3 AR(i) is never optimal. Therefore the optimal decision is either PP or AD(i). By Proposition G.3 the optimal decision is PP if and only if $z\leqslant\left(\frac{a+udr}{w(u-\widehat{u})}\right)\wedge\underline{z}_p(\gamma)$, which implies the optimal decision is AD(i) if and only if $z>\left(\frac{a+udr}{w(u-\widehat{u})}\right)\wedge\underline{z}_p(\gamma)$. Setting $\underline{z}=\bar{z}=\left(\frac{a+udr}{w(u-\widehat{u})}\right)\wedge\underline{z}_p(\gamma)$ establishes the property Proposition G.4 describes.

(*b*) $\frac{a+udr}{w(u-\widehat{u})} < \underline{z}_r(\gamma)$. By Proposition G.3 the optimal decision is AR(i) if and only if $\frac{a+udr}{w(u-\widehat{u})} < z \leqslant \underline{z}_r(\gamma)$. Suppose $\frac{a+udr}{w(u-\widehat{u})} \leqslant \underline{z}_p(\gamma)$ then the optimal decision is PP if and only if $z \leqslant \frac{a+udr}{w(u-\widehat{u})}$. So setting $\underline{z} = \frac{a+udr}{w(u-\widehat{u})}$ and $\bar{z} = \underline{z}_r(\gamma)$ will establish the property Proposition G.4 describes. We only need to show that indeed $\frac{a+ur}{uw} \leqslant \underline{z}_p(\gamma)$ under case (*b*).

By way of contradiction suppose $\frac{a+ur}{uw} > \underline{z}_p(\gamma)$. It implies that there exists $i'' \in U_\gamma$ such that

$$z_p(\gamma, i'') = \frac{a + ud\nabla(\gamma, i'')}{w\{ud[\widehat{u}(|S_{g_\gamma}| - |S_{g_{\gamma\ominus i''}}|) + (u-\widehat{u})(|U_\gamma| - |U_{\gamma\ominus i''}| - 1)] + (u-\widehat{u})]\}} < \frac{a+udr}{w(u-\widehat{u})}. \tag{G.132}$$

On the other hand $\frac{a+udr}{w(u-\widehat{u})} \leqslant \underline{z}_r(\gamma)$ implies

$$\frac{a+udr}{w(u-\widehat{u})} \leqslant z_r(\gamma, i'') = \frac{\nabla(\gamma, i'') - r}{w[(u-\widehat{u})(|U_\gamma| - |U_{\gamma\ominus i''}| - 1) + \widehat{u}(|S_{g_\gamma}| - |S_{g_{\gamma\ominus i''}}|)]}. \tag{G.133}$$

Let $M_1 = a + udr$, $N_1 = w(u-\widehat{u})$, $M_2 = ud(\nabla(\gamma, i) - r)$, and $N_2 = udw[\widehat{u}(|S_{g_\gamma}| - |S_{g_{\gamma\ominus i''}}|) + (u-\widehat{u})(|U_\gamma| - |U_{\gamma\ominus i''}| - 1)]$. Then $\frac{a+udr}{w(u-\widehat{u})} = \frac{M_1}{N_1}$, $z_r(\gamma, i'') = \frac{M_2}{N_2}$, and $z_p(\gamma, i'') = \frac{M_1+M_2}{N_1+N_2}$. Since $N_1, N_2 > 0$ and $\frac{M_1}{N_1} = \frac{a+udr}{w(u-\widehat{u})} < z_r(\gamma, i'') = \frac{M_2}{N_2}$, it must hold that $\frac{M_1}{N_1} < \frac{M_1+M_2}{N_1+N_2} < \frac{M_2}{N_2}$, i.e., $\frac{a+udr}{w(u-\widehat{u})} < z_p(\gamma, i'') < z_r(\gamma, i'')$, contradicting (G.132). □

We consider a state $\gamma_+ = (g, U)$ in which there is at least one supplier in each position in tier 2 (majority-exclusive, minority-exclusive, shared; i.e., $t_{\text{A}}, t_{\text{B}}, t_{\text{AB}} \geqslant 1$), all suppliers (including those in tier 1) are unaudited, and the majority tier-1 firm A has strictly more suppliers than the minority tier-1 firm B (i.e., $t_{\text{A}} > t_{\text{B}}$). This structure allows us to compare all possible auditing choices. In Proposition G.5 the thresholds $z_{\text{A}|\text{B}}$, $z_{\text{B}|1}$, and $z_{\text{A}|1}$ are identical to those in Proposition D.5.

Proposition G.5. *At state γ_+ suppose the buyer can audit at most one supplier before proceeding to production (* PP*). Let $\underline{z}_d = (z_{\text{A}|1} \wedge z_{\text{B}|1}) \vee \bar{z}$ and $\bar{z}_d = z_{\text{A}|1} \vee z_{\text{A}|\text{B}} \vee \bar{z}$ where $\bar{z}$ is as in Proposition G.4. The optimal decision is* AD(ⓔ_{A}) *(i.e., auditing and dropping (if noncompliant) an exclusive supplier to firm* A*) if and only if $\bar{z} < z \leqslant \underline{z}_d$,* AD(B) *if and only if $\underline{z}_d < z \leqslant \bar{z}_d$, and* AD(A) *if and only if $z > \bar{z}_d$.*

Proof. By the proof of Proposition G.3, let $i, i' \in U_\gamma$, $\widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i)) \geqslant \widetilde{V}(\xi_{\text{PP}}, \gamma, \text{AD}(i'))$ iff

$$\pi(\gamma \ominus i) - wz[\widehat{u}|S_{g_{\gamma\ominus i}}| + (u-\widehat{u})|U_{\gamma\ominus i}|] \geqslant \pi(\gamma \ominus i') - wz[\widehat{u}|S_{g_{\gamma\ominus i'}}| + (u-\widehat{u})|U_{\gamma\ominus i'}|]. \tag{G.134}$$

Since in γ_+, $|S_{g_{\gamma\ominus i}}| = |U_{\gamma\ominus i}|$, (G.134) is equivalent to (D.128). Therefore the rest of the proof of Proposition G.5 is identical to the proof of Proposition D.5. □

Proposition G.6. *Consider a nonterminal state where the buyer can audit at most one supplier before proceeding to production. As penalty z increases, the optimal action shifts from* PP *to* AR(i) *(where i is any unaudited supplier) to* AD(i) *(where i is a certain unaudited supplier). Further, if every supplier in the network is unaudited (and there is at least one supplier in each position in tier 2, and the majority tier-1 firm* A *has strictly more suppliers than the minority tier-1 firm* B*), within the interval of z where* AD(i) *is optimal, as z increases, the supplier i to audit shifts from* ⓔ_{A} *to* B *to* A.

Proof. Proposition G.6 is a summary of Propositions G.4 and G.5. □

Appendix H: The Production Profit Function Approach with General Supply Networks

We consider general supply networks without any restrictions over tiers or the number of suppliers within any tier. To handle the generality of the network we abstract away a specific model of production but instead adopt an abstract production profit function that encapsulates any underlying production activity. It represents the profit the buyer makes from production on a given supply network.

H.1. Model

H.1.1. Supply Network We model a general supply network with a single buyer and any finite number of suppliers. Each firm has at least one upstream supplier, except for raw materials suppliers. Every supplier in the network has at least one downstream customer which is either the buyer or another supplier. We do not impose any restrictions on the supply relationships between firms. For instance a supplier may sell to a customer but also directly to this customer's own customer at the same time. All material flows eventually end at the buyer.

We represent the *supply network* as a directed graph $g=(W_g,E_g)$, where the set of vertices W_g represents the firms (the buyer and suppliers) in the supply network and the set of arcs E_g represents the supply relationships between the firms. The direction of each arc in E_g represents the the flow of the goods: for $i,j\in W_g$, $ji\in E_g$ means "j supplies i". We denote the buyer by $\textsc{c}\in W_g$ and the set of suppliers by $S_g=W_g\backslash\{\textsc{c}\}$. The buyer $\textsc{c}$ is reachable from any $i\in S_g$. Each supplier i has outdegree $d_g^+(i)>0$. We denote G the set of all supply networks.

For $i,j\in S_g$, we call j a *dependent* of i in g if j solely relies on i to supply the buyer, i.e., i is on every directed path from a raw material supplier to the buyer $\textsc{c}$ that traverses j in g. Denote $D_g(i)$ the set of dependents of i in g. A dependent j of i may lie upstream or downstream from i. Two suppliers may simultaneously be dependents of each other. Every supplier is a dependent of itself, i.e., $i\in D_g(i),\forall i\in S_g$. We shall omit the subscript "g" whenever there is no risk of confusion.

H.1.2. Auditing We model the auditing phase as a Markov decision process for the buyer. A *state* consists of a supply network and the auditing status of each supplier (*unaudited* or *vetted*). Specifically a state is a tuple $\gamma=(g_\gamma,U_\gamma)$ where $g_\gamma=(W_{g_\gamma},E_{g_\gamma})$ is a supply network and $U_\gamma\subseteq S_{g_\gamma}$ is the set of suppliers that are currently *unaudited*. We omit the subscript "γ" whenever doing so causes no confusion. Any supplier $i\in S_g\backslash U$ is *vetted*. The *state space* is $\Gamma=\{(g,U):g\in G,U\subseteq S_g\}$. The terminal states Γ_T are the supply networks with no more unaudited suppliers, $\Gamma_T=\{\gamma=(g,U)\in\Gamma:U=\emptyset\}$.

As in our main model, to facilitate the formulation of the dynamic program, we define two operators that will be used when updating the state. Let $Z=\{(\gamma,i):\gamma\in\Gamma,i\in U_\gamma\}$ be the set of pairs of a state and an unaudited supplier (in that state). The first mapping $\oplus:Z\to\Gamma$ changes a supplier from an unaudited to a vetted status, i.e., given state γ and unaudited supplier i in γ, $\gamma\oplus i$ is the state otherwise identical to γ but with a vetted i.[14] The second mapping $\ominus:Z\to\Gamma$ removes a supplier along with its dependents from a state,

[14] Given $\gamma=(g,U)\in\Gamma$ and $i\in U_\gamma$, $\gamma\oplus i=(g,U')$ where $U'=U\backslash\{i\}$.

i.e., given state γ and unaudited supplier i in γ, $\gamma \ominus i$ is the state otherwise identical to γ but with i and all its dependents removed.[15]

Given $\gamma = (g, U) \in \Gamma$, we define a mapping $\pi : \Gamma \to \mathbb{R}$ so that $\pi(\gamma)$ is the buyer's *production profit* if the buyer chooses to stop auditing in state γ. It reflects the profit the buyer makes from the production activity on supply network g. We otherwise model the auditing activity in the same way as the auditing phase in our main model.

H.2. Results

THEOREM H.1. *There exists an optimal policy $\xi^* \in \Xi$ with the property that auditing decisions are divided into two subphases:*

(*a*) AD subphase: *To audit and drop (*AD*) some suppliers (or none); followed by*

(*b*) RP subphase: *To audit and rectify (*AR*) all remaining unaudited suppliers in an arbitrary sequence if $a + ur \leqslant uwz$; or to proceed to production (*PP*) if $a + ur \geqslant uwz$.*

COROLLARY H.1. *At state $\gamma \in \Gamma$, if the optimal policy ξ^* is already in the* RP *subphase,*

$$V^*(\gamma) = \pi(\gamma) - c_{\text{RP}}|U_\gamma|. \tag{H.1}$$

The proofs of Theorem H.1 and Corollary H.1 are identical to those of Theorem 3 and Corollary 1.

For the LVUS auditing result, the assumption of decreasing differences of the production profit remains the same as in our main model (Assumption D.1). Assumption D.2 (the preservation of LVUS) now requires a new definition of symmetric suppliers, as follows:

DEFINITION H.1 (SYMMETRIC SUPPLIERS). Let $\gamma = (g, U), \gamma' = (g', U') \in \Gamma$ be two (possibly identical) states. If there exist bijections $\theta : W_g \to W_{g'}$ and $\psi : E_g \to E_{g'}$ such that (θ, ψ) is an isomorphism between directed graphs g and g', and in addition $i \in U$ if and only if $\theta(i) \in U'$ for all $i \in S_g$, we call the pair (θ, ψ) an *isomorphism* between states γ and γ'. Let $\gamma = (g, U) \in \Gamma$ and $i, i' \in W_g$. We say i, i' are *symmetric* or i is symmetric with i' if there exists an isomorphism (θ, ψ) between γ and itself such that $\theta(i) = i'$.

THEOREM H.2. *Let $\gamma_0 = (g, U) \in \Gamma$ be such that for any $\gamma \in R^+(\gamma_0)$, no unaudited supplier in γ is a dependent of another unaudited supplier, i.e., any $i, i' \in U_\gamma$ $(i \neq i')$ satisfy $i \notin D_g(i')$ and $i' \notin D_g(i)$. Under Assumptions D.1 and D.2, the following policy ξ^* is optimal in every state $\gamma \in R^+(\gamma_0)$:*

$$\xi^*(\gamma) = \begin{cases} \text{AD}(i), & \text{if } i \in U_\gamma, u\nabla(\gamma, i) + a < c_{\text{RP}}, \text{ and } \nabla(\gamma, i) \leqslant \nabla(\gamma, i'), \forall i' \in U_\gamma \\ \text{RP}, & \text{if } u\nabla(\gamma, i) + a \geqslant c_{\text{RP}}, \forall i \in U_\gamma \end{cases}. \tag{H.2}$$

The proof of Theorem H.2 is identical to that of Theorem D.1.

[15] Given $\gamma = (g, U) \in \Gamma$ and $i \in U$, $\gamma \ominus i = (g', U')$ where $g' = g[W_g \backslash D_g(i)]$, the subgraph of g induced by $W_g \backslash D_g(i)$, and $U' = U \cap W_{g'}$.

Appendix I: Technical Lemmas

LEMMA I.1 **(determinant of upper arrowhead matrix)**. *For $n \in \mathbb{N}_+$ and $n \geqslant 2$, and $a, b, c, d \in \mathbb{R}$, then*

$$\det \begin{pmatrix} a & b & b & \cdots & b \\ c & d & 0 & \cdots & 0 \\ c & 0 & d & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c & 0 & \cdots & 0 & d \end{pmatrix}_{n\times n} = [ad - (n-1)bc]d^{n-2}. \tag{I.1}$$

Proof. Denote the matrix in (I.1) by Ψ. If $d = 0$ we expanding Ψ along the first column in the way of Laplace to find $\det(\Psi) = 0$ (each submatrix in the expansion has zero determinant). If $d > 0$ we multiply columns 2 to n each by $-\frac{c}{d}$ and add them all to the first column to get

$$\det(\Psi) = \det \begin{pmatrix} a - (n-1)\frac{bc}{d} & b & b & \cdots & b \\ 0 & d & 0 & \cdots & 0 \\ \vdots & & \ddots & d & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & \cdots & \cdots & 0 & d \end{pmatrix} \tag{I.2}$$

which, by the formula for the determinant of a diagonal matrix, is equal to the right-hand side of (I.1). □

For $m, n, k \in \mathbb{N}_0$, denote by $J_{m,n} = [1]_{m\times n}$ the $m \times n$ matrix of 1's; $J_n = J_{n,n}$. Denote

$$D_n = I_n + J_n = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}_{n\times n} \quad Q_{2m,n} = \begin{bmatrix} 1 & \cdots & 1 \\ 2 & \cdots & 2 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \\ 2 & \cdots & 2 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \\ 2 & \cdots & 2 \end{bmatrix}_{2m\times n} \quad R_{2m,n} = \begin{bmatrix} 2 & \cdots & 2 \\ 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 2 & \cdots & 2 \\ 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 2 & \cdots & 2 \\ 1 & \cdots & 1 \end{bmatrix}_{2m\times n}. \tag{I.3}$$

Denote $\widetilde{D}_{2n} = D_n \otimes D_2$, where $\otimes$ denotes the Kronecker product. Finally, define the two $(m+2k+n) \times (m+2k+n)$ symmetric matrices

$$\Lambda(m,n,k) = \begin{bmatrix} 2D_m & R^\mathsf{T}_{2k,m} & J_{m,n} \\ R_{2k,m} & \widetilde{D}_{2k} & Q_{2k,n} \\ J_{n,m} & Q^\mathsf{T}_{2k,n} & 2D_n \end{bmatrix}, \quad \widetilde{\Lambda}(m,n,k) = \begin{bmatrix} 2D_{m+k} & \begin{bmatrix} J_{m,k} & J_{m,n} \\ D_k & J_{k,n} \end{bmatrix} \\ \begin{bmatrix} J_{k,m} & D_k \\ J_{n,m} & J_{n,k} \end{bmatrix} & 2D_{k+n} \end{bmatrix}. \tag{I.4}$$

LEMMA I.2. *For m, n, k such that $m + n + k > 0$, $\widetilde{\Lambda}(m,n,k)$ is positive definite.*

Proof. Denote $m' = m + k$ and $n' = n + k$. Denote

$$B = \begin{bmatrix} J_{m,k} & J_{m,n} \\ D_k & J_{k,n} \end{bmatrix} \text{ so that } \widetilde{\Lambda} = \begin{bmatrix} 2D_{m'} & B \\ B^\mathsf{T} & 2D_{n'} \end{bmatrix}. \tag{I.5}$$

The proof consists of two parts:

(a) $2D_{m'}$ is positive definite;

(b) The Schur complement of $2D_{m'}$ in $\widetilde{\Lambda}$, i.e., $\widetilde{\Lambda}/(2D_{m'}) = 2D_{n'} - B^\mathsf{T}(2D_{m'})^{-1}B$, is positive definite.

Then by the Schur complement condition (Boyd and Vandenberghe 2004, Appendix A.5.5), $\widetilde{\Lambda}$ is positive definite, as we want.

(a) To show that $2D_{m'}$ is positive definite, it suffices to show that every leading principal minor of $D_{m'}$ is positive. For $i \in \mathbb{N}_+$, subtract the first row of D_i from every other row to get

$$\det(D_i) = \det\begin{pmatrix} 2 & 1 & \cdots & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{I.6}$$

which by Lemma I.1 equals $(2\times 1-(i-1)\times 1\times(-1))\times 1^{i-2} = i+1>0$. Therefore $2D_{m+k}$ is positive definite.

(b) We next show that $2D_{n'} - B^{\mathsf{T}}(2D_{m'})^{-1}B$ is positive definite. Use row reduction to find

$$(2D_{m'})^{-1} = \frac{1}{2(m'+1)}\begin{bmatrix} m' & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & m' \end{bmatrix}. \tag{I.7}$$

Then

$$B^{\mathsf{T}}(2D_{m'})^{-1}B = \begin{bmatrix} J_{k,m} & D_k \\ J_{n,m} & J_{n,k} \end{bmatrix}\left(\frac{1}{2(m'+1)}\begin{bmatrix} m' & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & m' \end{bmatrix}\right)\begin{bmatrix} J_{m,k} & J_{m,n} \\ D_k & J_{k,n} \end{bmatrix} \tag{I.8}$$

$$= \frac{1}{2(m'+1)}\underbrace{\begin{bmatrix} 1 & \cdots & 1 & 2 & 1 & \cdots & 1 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 1 \\ \vdots & & & & & \ddots & 2 \\ \vdots & & & & & & 1 \\ \vdots & & & & & & \vdots \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}}_{n'\times m'}\underbrace{\begin{bmatrix} m' & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & m' \end{bmatrix}}_{m'\times m'}\begin{bmatrix} J_{m,k} & J_{m,n} \\ D_k & J_{k,n} \end{bmatrix} \tag{I.9}$$

$$= \frac{1}{2(m'+1)}\underbrace{\begin{bmatrix} 0 & \cdots & 0 & m'+1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots\cdots & \cdots & & \cdots & 0 & m'+1 \\ 1 & \cdots\cdots & \cdots & & \cdots\cdots & & 1 \\ \vdots & & & & & & \vdots \\ 1 & \cdots\cdots & \cdots & & \cdots\cdots & & 1 \end{bmatrix}}_{n'\times m'}\underbrace{\begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & & & & & & \vdots \\ 1 & & & & & & \vdots \\ 2 & \ddots & & & & & \vdots \\ 1 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 1 & \cdots & 1 & 2 & 1 & \cdots & 1 \end{bmatrix}}_{m'\times n'} \tag{I.10}$$

$$
= \frac{1}{2(m'+1)} \underbrace{\begin{bmatrix}
2(m'+1) & m'+1 & \cdots & \cdots & m'+1 & m'+1 & \cdots & m'+1 \\
m'+1 & \ddots & \ddots & & \vdots & \vdots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\
\vdots & & \ddots & \ddots & m'+1 & \vdots & & \vdots \\
m'+1 & \cdots & \cdots & m'+1 & 2(m'+1) & m'+1 & \cdots & m'+1 \\
m'+1 & \cdots & \cdots & \cdots & m'+1 & m' & \cdots & m' \\
\vdots & & & & \vdots & \vdots & & \vdots \\
m'+1 & \cdots & \cdots & \cdots & m'+1 & m' & \cdots & m'
\end{bmatrix}}_{n' \times n'} \tag{I.11}
$$

$$
= \underbrace{\begin{bmatrix}
1 & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \ddots & \ddots & & \vdots & \vdots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\
\vdots & & \ddots & \ddots & \frac{1}{2} & \vdots & & \vdots \\
\frac{1}{2} & \cdots & \cdots & \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \cdots & \cdots & \cdots & \frac{1}{2} & \frac{m'}{2(m'+1)} & \cdots & \frac{m'}{2(m'+1)} \\
\vdots & & & & \vdots & \vdots & & \vdots \\
\frac{1}{2} & \cdots & \cdots & \cdots & \frac{1}{2} & \frac{m'}{2(m'+1)} & \cdots & \frac{m'}{2(m'+1)}
\end{bmatrix}}_{n' \times n'}. \tag{I.12}
$$

Therefore

$$
2D_{n'} - B^{\mathsf{T}}(2D_{m'})^{-1}B = \underbrace{\begin{bmatrix}
3 & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} \\
\frac{3}{2} & \ddots & \ddots & & \vdots & \vdots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
\vdots & & \ddots & \ddots & \frac{3}{2} & \vdots & & & \vdots \\
\frac{3}{2} & \cdots & \cdots & \frac{3}{2} & 3 & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} \\
\frac{3}{2} & \cdots & \cdots & \cdots & \frac{3}{2} & \frac{7m'+8}{2(m'+1)} & \frac{3m'+4}{2(m'+1)} & \cdots & \frac{3m'+4}{2(m'+1)} \\
\vdots & & & & \vdots & \frac{3m'+4}{2(m'+1)} & \ddots & \ddots & \vdots \\
\vdots & & & & \vdots & \vdots & \ddots & \ddots & \frac{3m'+4}{2(m'+1)} \\
\frac{3}{2} & \cdots & \cdots & \cdots & \frac{3}{2} & \frac{3m'+4}{2(m'+1)} & \cdots & \frac{3m'+4}{2(m'+1)} & \frac{7m'+8}{2(m'+1)}
\end{bmatrix}}_{n' \times n'}. \tag{I.13}
$$

It only remains to show that (I.13) is positive definite. We conduct the following row operations on the matrix so that each newly formed row i only references the rows $j \leqslant i$ in (I.13), ensuring that the leading principal minors are preserved:

$$
\text{(I.13)} \xrightarrow[\forall i \geqslant 2]{R_i - R_1} \begin{bmatrix}
3 & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} \\
-\frac{3}{2} & \frac{3}{2} & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & & & \vdots \\
-\frac{3}{2} & 0 & \cdots & 0 & \frac{3}{2} & 0 & \cdots & \cdots & 0 \\
-\frac{3}{2} & 0 & \cdots & \cdots & 0 & \frac{4m'+5}{2(m'+1)} & \frac{1}{2(m'+1)} & \cdots & \frac{1}{2(m'+1)} \\
\vdots & \vdots & & & \vdots & \frac{1}{2(m'+1)} & \ddots & \ddots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & \ddots & \ddots & \frac{1}{2(m'+1)} \\
-\frac{3}{2} & 0 & \cdots & \cdots & 0 & \frac{1}{2(m'+1)} & \cdots & \frac{1}{2(m'+1)} & \frac{4m'+5}{2(m'+1)}
\end{bmatrix} \tag{I.14}
$$

$$\xrightarrow[\forall i\in[k+2,k+n]]{R_i-R_{k+1}}\begin{bmatrix} 3 & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} & \frac{3}{2} & \cdots & \cdots & \cdots & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & & & \vdots \\ -\frac{3}{2} & 0 & \cdots & 0 & \frac{3}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{3}{2} & 0 & \cdots & \cdots & 0 & \frac{4m'+5}{2(m'+1)} & \frac{1}{2(m'+1)} & \cdots & \cdots & \frac{1}{2(m'+1)} \\ 0 & \vdots & & & \vdots & -2 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & -2 & 0 & \cdots & 0 & 2 \end{bmatrix}. \tag{I.15}$$

To show that (I.13) is positive definite, it suffices to show that every leading principal minor $\det(M_i)$ of (I.15) is positive.

For $i=1,\ldots,k$, by Lemma I.1,

$$\det(M_i)=\left(3\times\frac{3}{2}-(i-1)\times\frac{3}{2}\times(-\frac{3}{2})\right)\times\left(\frac{3}{2}\right)^{i-2}=\left(\frac{3}{2}\right)^{i}(i+1)>0. \tag{I.16}$$

For $i=k+1$, (i) expanding

$$M_{k+1}=\begin{bmatrix} 3 & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ -\frac{3}{2} & 0 & \cdots & 0 & \frac{3}{2} & 0 \\ -\frac{3}{2} & 0 & \cdots & \cdots & 0 & \frac{4m'+5}{2(m'+1)} \end{bmatrix} \tag{I.17}$$

in the way of Laplace along the last row, then (ii) expanding the $(k+1,1)$th minor of M_{k+1} along the last column and (iii) applying Lemma I.1 on the $(k+1,k+1)$th minor of M_{k+1}, we get

$$\begin{aligned}\det(M_{k+1})=&(-1)^{k+2}\left(-\frac{3}{2}\right)(-1)^{(k+1)}\frac{3}{2}\left(\frac{3}{2}\right)^{k-1}\\ &+(-1)^{2(k+1)}\frac{4m'+5}{2(m'+1)}\left[(3)\left(\frac{3}{2}\right)-(k-1)\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)\right]\left(\frac{3}{2}\right)^{k-2}.\end{aligned} \tag{I.18}$$

It simplifies to

$$\det(M_{k+1})=\left(\frac{3}{2}\right)^{k+1}+\frac{4m'+5}{2(m'+1)}(k+1)\left(\frac{3}{2}\right)^{k}>0. \tag{I.19}$$

For $i=k+2,\ldots,k+n$, expand M_i along the $(k+1)$th row:

$$\det(M_i)=(-1)^{k+2}\left(-\frac{3}{2}\right)\det(S_1)+(-1)^{2(k+1)}\frac{4m'+5}{2(m'+1)}\det(S_{k+1})+\frac{1}{2(m'+1)}\sum_{j=k+2}^{i}(-1)^{k+1+j}\det(S_j) \tag{I.20}$$

where S_j is the submatrix of M_i formed by deleting row $k+1$ and column j. We next evaluate the $\det(S_j)$.

(Evaluate $\det(S_1)$). Denote $S_1^{(l)}$ for $l = 1, \dots, k-1$ the $(i-1-l) \times (i-1-l)$ submatrix of S_1 obtained by deleting rows $2, \dots, l+1$ and columns with the same indices. In addition, let $S_1^{(0)} = S_1$. That is

$$S_1^{(l)} = \begin{bmatrix} \frac{3}{2} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{3}{2} \\ \frac{3}{2} & & & & & & \\ \frac{3}{2} & & & & & & \\ & \ddots & & & & & \\ & & \frac{3}{2} & & & & \\ & & & -2 & 2 & & \\ & & & \vdots & & \ddots & \\ & & & -2 & & & 2 \end{bmatrix} \tag{I.21}$$

where empty space represents zeros and there are $k-1-l$ entries of $\frac{3}{2}$ on the subdiagonal and $i-k-1$ entries of 2 on the diagonal. In particular, there is no $\frac{3}{2}$ on the subdiagonal of $S_1^{(k-1)}$ since all of them are deleted. Then expand $S_1^{(l)}$ successively along the second row:

$$\det(S_1) = \det(S_1^{(0)}) = -\frac{3}{2}\det(S_1^{(1)}) = \cdots = \left(-\frac{3}{2}\right)^{l} \det(S_1^{(l)}) = \cdots = \left(-\frac{3}{2}\right)^{k-1} \det(S_1^{(k-1)}) \tag{I.22}$$

$$= \left(-\frac{3}{2}\right)^{k-1} \left[\left(\frac{3}{2}\right)(2) - (i-k-1)\left(\frac{3}{2}\right)(-2)\right](2)^{i-k-2} = (-1)^{k-1}\left(\frac{3}{2}\right)^{k}(i-k)(2)^{i-k-1}. \tag{I.23}$$

(Evaluate $\det(S_{k+1})$).

$$S_{k+1} = \begin{bmatrix} 3 & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} & \frac{3}{2} & \cdots & \cdots & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & & \vdots \\ -\frac{3}{2} & 0 & \cdots & 0 & \frac{3}{2} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 2 \end{bmatrix}_{(i-1)\times(i-1)} \equiv \begin{bmatrix} A_{k\times k} & E_{k\times(i-k-1)} \\ 0_{(i-k-1)\times k} & 2I_{i-k-1} \end{bmatrix}. \tag{I.24}$$

The submatrix A in the upper-left block is an arrowhead matrix; by Lemma I.1,

$$\det(A) = \left[(3)\left(\frac{3}{2}\right) - (k-1)\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)\right]\left(\frac{3}{2}\right)^{k-2} = (k+1)\left(\frac{3}{2}\right)^{k}. \tag{I.25}$$

Observing that the lower-left block of the matrix is a submatrix of zeros, we find

$$\det(S_{k+1}) = \det(A)\det(2I_{i-k-1}) = (k+1)\left(\frac{3}{2}\right)^{k} 2^{i-k-1}. \tag{I.26}$$

(Evaluate $\det(S_j), \forall j > k+1$). For $j > k+1$,

$$S_j = \begin{bmatrix} A_{k\times k} & E_{k\times(i-k-1)} \\ 0_{(i-k-1)\times k} & Y_j \end{bmatrix}_{(i-1)\times(i-1)} \tag{I.27}$$

where A and E are the same as in (I.24) and

$$Y_j = \begin{bmatrix} -2 & 2 & & & & & \\ \vdots & & \ddots & & & & \\ \vdots & & & 2 & & & \\ \vdots & & & 0 & 0 & & \\ \vdots & & & & 2 & & \\ \vdots & & & & & \ddots & \\ -2 & & & & & & 2 \end{bmatrix}_{(i-k-1)\times(i-k-1)} \tag{I.28}$$

where, in particular, the $(j-k+1)$th row is $(-2,0,\dots,0)$. Expand its determinant along this row:

$$\det(Y_j) = (-1)^{j-k+2}(-2)\det(2I_{i-k-2}) = (-1)^{j-k+3}(2^{i-k-1}). \tag{I.29}$$

Since the lower-left submatrix of S_j is zero, we have

$$\det(S_j) = \det(A)\det(Y_j) = (-1)^{j-k+3}(k+1)\left(\frac{3}{2}\right)^k (2^{i-k-1}) \tag{I.30}$$

where (I.25) is substituted for $\det(A)$.

Substitute (I.23), (I.26), and (I.30) into (I.20) to get

$$\det(M_i) = \left(\frac{3}{2}\right)^{k+1}(i-k)(2)^{i-k-1} + \frac{4m'+5}{2(m'+1)}(k+1)\left(\frac{3}{2}\right)^k 2^{i-k-1} + \frac{1}{2(m'+1)}\sum_{j=k+2}^{i}(k+1)\left(\frac{3}{2}\right)^k(2^{i-k-1}) > 0. \tag{I.31}$$

Therefore (I.15) is positive definite. The proof is complete. □

LEMMA I.3. *For m,n,k such that $m+n+k>0$, $\Lambda(m,n,k)$ is positive definite.*

Proof. In the following we omit the arguments (m,n,k) of the matrices Λ and $\widetilde{\Lambda}$ as doing so causes no confusion.

Rewrite $\widetilde{\Lambda}$ in (I.4) by breaking up its upper-left and lower-right blocks as

$$\widetilde{\Lambda}(m,n,k) = \begin{bmatrix} 2D_m & 2J_{m,k} & J_{m,k} & J_{m,n} \\ 2J_{k,m} & 2D_k & D_k & J_{k,n} \\ J_{k,m} & D_k & 2D_k & 2J_{k,n} \\ J_{n,m} & J_{n,k} & 2J_{n,k} & 2D_n \end{bmatrix}_{(m+2k+n)\times(m+2k+n)}. \tag{I.32}$$

Now the blocks in the four corners of $\widetilde{\Lambda}$ are identical to the corresponding blocks in Λ. The idea is to permute the rows and columns of Λ in between the corner blocks symmetrically to obtain $\widetilde{\Lambda}$. Specifically, let P be the $(m+2k+n)\times(m+2k+n)$ permutation matrix that leaves rows $1,\dots,m$ and $m+2k+1,\dots,m+2k+n$ intact and permutes rows $m+1,\dots,m+2k$ according to the permutation of the set $\{1,\dots,2k\}$ (where element l corresponds to row $m+l$) as follows (in Cauchy's two-line notation):

$$\begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & k-1 & k & k+1 & k+2 & \cdots & j & j+1 & \cdots & 2k-1 & 2k \\ 1 & k+1 & \cdots & i & k+i & \cdots & k-1 & 2k-1 & 2 & k+2 & \cdots & j-k+1 & j+1 & \cdots & k & 2k \end{pmatrix} \tag{I.33}$$

where i is any odd number between 1 and $k-1$ and j is any odd number between $k+1$ and $2k-1$ (all ends inclusive), for k even, and

$$\begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & k-2 & k-1 & k & k+1 & k+2 & \cdots & j & j+1 & \cdots & 2k-2 & 2k-1 & 2k \\ 1 & k+2 & \cdots & i & k+i+1 & \cdots & k-2 & 2k-1 & k & 2 & k+3 & \cdots & j-k+1 & j+2 & \cdots & k-1 & 2k & k+1 \end{pmatrix} \tag{I.34}$$

where i is any odd number between 1 and $k-2$ and j is any even number between $k+1$ and $2k-2$ (all ends inclusive), for k odd. Then $\widetilde{\Lambda} = P\Lambda P^\mathsf{T}$.

By Lemma I.2, $\widetilde{\Lambda}$ is positive definite. By Theorem 6C(V) in Strang (1980), there exists an invertible matrix Q such that $\widetilde{\Lambda} = Q^\mathsf{T}Q$. Since P is a permutation matrix, the matrix $Q(P^{-1})^\mathsf{T}$ is invertible. Therefore, again by Theorem 6C(V) in Strang (1980), $(Q(P^{-1})^\mathsf{T})^\mathsf{T}Q(P^{-1})^\mathsf{T}$ is positive definite. But since $\widetilde{\Lambda} = P\Lambda P^\mathsf{T}$,

$$(Q(P^{-1})^\mathsf{T})^\mathsf{T}Q(P^{-1})^\mathsf{T} = P^{-1}Q^\mathsf{T}Q(P^{-1})^\mathsf{T} = P^{-1}\widetilde{\Lambda}(P^\mathsf{T})^{-1} = P^{-1}P\Lambda P^\mathsf{T}(P^\mathsf{T})^{-1} = \Lambda \tag{I.35}$$

implying Λ is positive definite. □

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