Theorem 2 Assume $p_2 > h_2(1 - \beta)/\beta$. Let salvage value functions \( S_0^1(\tilde{X}) = \lambda_1(X^1 - \gamma_1)^2 \) and \( S_0^2(\tilde{X}) = \lambda_2(\gamma_2 - X^2)^+ \). There exist \( \lambda_1, \lambda_2, \gamma_1, \) and \( \gamma_2 \) such that for each starting inventory \( \tilde{X} \in B_k \):

(i) using Pareto refinement, there exists a unique pure-strategy Nash equilibrium, which is a modified echelon base-stock policy, in period \( k \);

(ii) \( V_k^1(\tilde{X}) = V_k^{1,1}(X^1) + V_k^{2}(X^2) \) for \( j = 1, 2 \); and

(iii) \( z_k^j \leq z_{k+1}^j \) for \( j = 1, k \geq 1 \) and \( j = 2, k \geq 2 \).

Proof This proof is by induction. In addition to showing that the main conditions (i)-(iii) hold, we also use the following conditions as a part of the inductive loop for \( \tilde{Y} \in B_k \): (iv) \( J_{k+1}^i \) is convex in \( Y^1 \) for \( Y^1 \leq z_{k+1}^i \); (v) \( J_{k+1}^i \) is convex decreasing in \( Y^2 \leq z_{k+1}^2 \), and decreasing in \( Y^2 \geq z_{k+1}^2 \); (vi) \( J_{k+1}^2 \) is convex in \( Y^2 \); (vii) \( J_{k+1}^1 \) is convex decreasing in \( Y^1 \leq (\min(z_{k+1}^i - K_1, z_{k+1}^j), z_{k+1}^1) \), and decreasing in \( Y^1 \geq (\min(z_{k+1}^i - K_1, z_{k+1}^j), z_{k+1}^1) \); (viii) \( J_{k+1}^i(Y^1, Y^2) = J_{k+1}^1(Y^1) + J_{k+1}^2(Y^2) \) for \( i \in \{1, 2\} \) and \( z_{k+1}^i := \arg \min_{Y^1} J_{k+1}^i(Y^1) \); (ix) \( \delta_{ij} J_{k+1}^i(\tilde{Y}) \geq \delta_{ij} J_{k+1}^i(\tilde{Y}) \) for \( Y^1 \leq z_{k+1}^i \) when \((i, j) \in \{(1, 1), (1, 2), (2, 2)\}\) and for \( Y^1 \leq \min(z_{k+1}^i - K_1, z_{k+1}^i) \) when \((i, j) = (2, 1)\); (x) \( z_{k+1}^i = y_{\ast y}^i \) and outside the band (xii) \( \forall Y \geq Y^1 + K_1, J_{k+1}^i \) is convex in \( Y^1 \), \( J_{k+1}^2 \) is increasing in \( Y^2 \), \( \partial_{ij} J_{k+1}^i \leq 0 \) for \( Y^1 + K_1 \leq Y^2 \leq z_{k+1}^i + 2K_1, Y^1 \leq z_{k+1}^i \}, \partial_{ij} J_{k+1}^i = 0 \) for \( Y^1 + K_1 \leq Y^2 \leq z_{k+1}^i \), and for \( \tilde{Y} \in B^{K_1} \), we have \( J_{k+1}^i(\tilde{Y}) = J_{k+1}^i(Y^1) \) for \( \tilde{Y} \leq z_{k+1}^i \) and \( J_{k+1}^i(Y^1) = z_{k+1}^i \) for \( Y^1 \leq z_{k+1}^i + K_1 \), \( z_{k+1}^i + 2K_1 \) for \( Y^2 \leq Y^1 + K_1 \), and \( \partial_{ij} J_{k+1}^i(Y^2) \) for \( Y^2 \geq Y^1 + K_1 \), \( z_{k+1}^i \) for \( \forall Y^2 \geq Y^1 + K_1 \), \( z_{k+1}^i + 2K_1 \) for \( Y^2 \geq Y^1 + K_1 \), \( z_{k+1}^i + 2K_1 \) for \( Y^2 > z_{k+1}^i + 2K_1 \).

Some of these conditions correspond to the conditions in the statement of Lemma 2: inductive statements (iv) and (vi) correspond to Lemma 2’s condition (a); statement (viii) corresponds to condition (b); and inductive statement (xii) corresponds to condition (d). Conditions (v), (vii), and (xii) warrant special comment. The firms’ cost functions are decreasing (weakly) in the other firm’s inventory level since, when the retailer holds more inventory, she reduces the supplier’s cost of the consumer backlog: when the supplier holds more inventory, he reduces the chance of the retailer being starved of material. Both of these circumstances result in reduced costs for the other firm. (xii) means that firm 1’s best reply function rises vertically along the \( X^2 \) dimension for a height of \( K_1 \) above the band \( B \) and for greater values of \( Y^2 \) it will never return to the band, thus eliminating the possibility of additional equilibria at higher supplier stocking levels.

We start with the inductive step to clearly present the logic of the critical elements of the proof. In order for the inductive step to hold, one of the elements is condition (iii) that the “up-to” values, \( z_{k+1}^i \), are decreasing in the number of remaining periods \( n \). The initial steps for \( n = 1, 2 \) will make it possible - we present them after the inductive step, since they use mostly the same logic.

Induction step, Period \( n \): Assume (i) -(xiiii) above with index \( n - 1 \) replacing \( k \). Note from (viii), \( J^i_n \) separates \( (J^i_n(Y) = J^i_n(Y^1) + J^2_n(Y^2)) \) which immediately delivers a proper definition of \( z_{n+1}^2 = \arg \min_{Y^2} J^2_n(Y^2) \), due to the convexity of \( J^2_n \).

---

1 \( J^{i1} \) and \( J^{i2} \) do not have the same values in the band \( B \) and in \( B^{K_1} \).

2 In Definition 3, \( z_n^i \) was defined as not dependent on \( Y^i \).
Unconstrained response functions:
Each of the three boxes in Figure 2 illustrates the feasible set, $\mathcal{A}(\hat{X}) = \times_{j=1}^{2}\mathcal{A}^j(\hat{X})$. Note that echelon 2’s controllable costs are minimized at $z^2_n$ and that echelon 2 has no incentive to store more than $K_1$; that is, the supplier’s best response function remains in the band. Consequently, from Lemma 2, the unconstrained best-reply function (defined as in Lemma 2, that is ignoring the current period initial constraints while accounting for the discounted future expected costs) is

$$r^2_n = z^2_n[|Y^1, Y^1 + K_1|].$$

The retailer may be better off choosing actions outside of band $\mathcal{B}$ and consequently, her response function may depart from it. If the response function is limited to the band,

$$r^1_n = z^1_n[|Y^2 - K_1, Y^2|].$$

From induction assumption (xiii), $r^1_n = z^1_n$ for $z^1_n + K_1 \leq Y^2 \leq z^1_n + 2K_1$ and, from (xii), $r^1_n$ is within the band for $Y^2 \leq z^1_n + K_1$. Thus, from Lemma 2, $r^1_n = \min\{z^1_n[|Y^2 - K_1, Y^2|], z^1_n\}$ for $Y^2 \leq z^1_n + K_1$. Also, note that $z^*_n = y^*_my$ from induction assumption (xi).

Unconstrained equilibria:

We consider first the unconstrained response functions and unconstrained equilibrium, which ignore the capacity constraints. The conditions of Theorem 1 (strategy spaces are nonempty compact convex subsets of Euclidean space, payoff functions are continuous and quasi-convex in $Y^1$) are satisfied for a given starting inventory position, $\hat{X} \in \mathcal{B}$; there exists at least one pure strategy equilibrium. As illustrated in Figure 1(C), a unique equilibrium exists, if $y^*_my = z^*_n \leq z^2_n \leq z^1_n + K$, for $n \geq 2$. Otherwise, the unconstrained response functions overlap over a range $[Y^1, Y^1 + K_1]$ for $y^*_my \leq Y^1 \leq \min(z^1_n, z^2_n - K_1)$, resulting in multiple equilibria. Recall that Pareto refinement discards all equilibria with higher costs for both players. Here, the use of Pareto refinement results in a single undominated equilibrium: Clearly conditions (v) and (vii) apply for the multiple-equilibria interval and, thus, $J^1_n$ is non-increasing in $Y^{-i}$ ($Y^{-i}$ refers to the player who is not $i$), implying that there exists an equilibrium with lower costs for each of the players, which is $(Y^1, Y^1 + K_1)$, where $Y^1 = \min(z^1_n, z^2_n - K_1)$.

Constrained equilibria:

Given the unconstrained equilibria, the properties (monotonicities) of individual value functions, $J^1_n$, and of response functions, we now describe the constrained equilibria (actual equilibria given the initial state). We use the three possible cases, shown in Figure 1, in which these best-reply functions can interact: (A) $z^1_n \leq z^2_n - K_1$, (B) $z^1_n < z^2_n - K_1 \leq z^1_n$, and (C) $z^2_n - K_1 \leq z^1_n \leq z^2_n \leq z^1_n + K_1$. In the case when $z^2_n < z^1_n$, the dynamics are very similar to the extreme case of $z^2_n = z^1_n$ within case (C). Since the logic is the same, we do not explicitly consider this case here. The restrictions on $Y^1$ in induction statement (vii) are intended to guarantee that all the properties are valid for $Y^1 \leq \min(z^2_n - K_1, z^1_n)$, relating to Figure 1(A)(B). However, in case (C), $z^1_n$ defines the equilibrium. Therefore, all properties are proved in the area up to the equilibrium. Considering case (A) first, we find the equilibrium: $(z^1_n||X^1, X^2, z^2_n||X^1, X^2 + K_1)$. Likewise for case (B): $(z^2_n - K_1)||X^1, X^2, (z^2_n - K_1)||X^1, X^2 + K_1)$. For case (C): $(z^1_n||X^1, X^2, z^2_n||X^2, X^2 + K_1)$. We now show (i), (ii), (iv), (v), (vi), (vii):

The Pareto dominating equilibrium is clearly a modified echelon base-stock policy, $Y^1_n = \min(z^1_n, z^2_n - K_1)||X^1, X^2_n)$ for cases (A) and (B), $Y^1_n = z^1_n||X^1, X^2_n$ for case (C), and $Y^2_n = z^2_n||X^2, Y^1_n + K_1$, demonstrating (i). Due to assumed convexities and separabilities, now we show that $V^*_n$ is separable, $J^*_n+1$ is convex in $Y^1$, and $J^1_n+1$ is convex in $Y^2 \leq z^2_n$ and $J^2_n+1$ is convex for $Y^1 \leq \min(z^2_n - K_1, z^1_n)$.

Consider case (A). Since $J^1_n$ is convex in $Y^1$ and convex decreasing in $Y^2 \leq z^2_n$, the function $J^1_n(Y^1, Y^1 + K_1)$ is convex for $Y^1 \leq z^2_n - K_1$, minimized at $Y^1 = z^1_n$, where $z^1_n \leq z^2_n - K_1$. Point
(ii) for case (A) follows, as the equilibrium value function depends only on \(Y^1\), for \(X^2 \leq z_n^2 - K_1\). Due to the feasibility constraints, \(Y^1 = X^2\) below the equilibrium up-to levels. In order to show properties of \(J_n^{i+1}\), it is sufficient to show them for functions \(V_n^i\): \(L^i\) is convex and separable (not influencing \(J_n^{i+1}\)) and the demand operator shifts \(\beta E[V_n^i(\tilde{Y} - \tilde{D})]\) downwards, so adding these two components results in all desired properties for \(J_n^{i+1}\). \(V_n^i\) is flat (constant valued) in \(X^1 \leq z_n^1\) and convex increasing in \(z_n^2 \leq X^2\) due to the following.

Note that \(\partial_x V_n^i(\tilde{X}) = DJ_n^i(Y^1)|_{Y^1 = X^1} + DJ_n^2(Y^2)|_{Y^2 = X^2 + K_1}\) for \(z_n^1 \leq X^1 \leq z_n^2 - K_1\) and \(\partial_x V_n^i(\tilde{X}) = DJ_n^i(Y^1)|_{Y^1 = X^1}\) for \(z_n^2 - K_1 \leq X^1\). Since \(DJ_n^i(Y^1)|_{Y^2 = X^1 + K_1} \leq 0\) for \(z_n^1 \leq X^1 \leq z_n^2 - K_1\) and both terms are non-decreasing, convexity follows thus implying (iv). \(V_n^i\) is convex decreasing in \(X^1 \leq z_n^2\) and flat in \(z_n^1 \leq X^2 \leq z_n^2\), and \(V_n^i\) is decreasing in \(X^2 \geq z_n^2\) from inductive assumption (v) for \(k = n - 1\), implying (v) for \(k = n\). \(V_n^2\) is convex decreasing in \(X^2 \leq z_n^2\) and flat in \(z_n^1 \leq X^2 \leq z_n^2\) and convex increasing in \(X^2 \geq z_n^2\). \(V_n^2\) is flat in \(X^1 \leq z_n^1\), decreasing in \(z_n^1 \leq X^1\), implying (vi) and (vii). The logic is similar for case (B). \(V_n^i\) is flat in \(X^1 \leq z_n^2 - K_1\) and convex increasing in \(X^1 \geq z_n^2 - K_1\) (implying (iv)). \(V_n^4\) is convex decreasing in \(X^2 \leq z_n^2 - K_1\), flat in \(z_n^2 - K_1 \leq X^2 \leq z_n^2\) and decreasing in \(X^2 \geq z_n^2\) (implying (v)). \(V_n^2\) is convex decreasing in \(X^2 \leq z_n^2 - K_1\), flat in \(z_n^2 - K_1 \leq X^2 \leq z_n^2\) and convex increasing in \(X^2 \geq z_n^2\) (justifying (vi)). \(V_n^2\) is flat in \(X^1 \leq z_n^1\), decreasing for \(X^1 > z_n^1\), convex decreasing for \(X^2 \leq z_n^2 - K_1\) and convex increasing in \(X^2 > z_n^2 - K_1\) (justifying (vii)). Similarly, for case (C) \(V_n^i\) is flat for \(X^1 \leq z_n^1\), convex increasing for \(X^1 > z_n^1\), convex decreasing for \(X^2 \leq z_n^2 - K_1\), and decreasing in \(X^2 > z_n^2 - K_1\) (justifying (vii)). Also, \(V_n^2\) is flat for \(X^1 \leq z_n^1\), decreasing for \(X^1 > z_n^1\), convex decreasing for \(X^2 \leq z_n^2 - K_1\) and convex increasing in \(X^2 > z_n^2 - K_1\) (justifying (vii)). Note that \(J_n^i\) is separable from induction assumption (ii) and the separability of \(L^i\). The separability of \(V_n^i\) follows due to the fact that the equilibrium in period \(n\) depends upon \(X^1\) or \(X^2\) but not a combination of both within a single parameter. This shows (ii).

- Ordering of derivatives and up-to levels:

  Let us define the following parameterized reference inequalities, which will permit analysis of the various subcases, using the indices \((i, j)\), where \(i \in \{1, 2\}\) denotes firm \(i\) and the derivative variable is \(X^j\), \(j \in \{1, 2\}\). These reference inequalities will be shown later for specific ranges of \(X^j\).

\[
\partial_x V_n^i(\tilde{X}) = 0 = \partial_x V_n^{i-1}(\tilde{X}) \quad (Eq1)
\]

\[
\partial_x V_n^i(\tilde{X}) = DJ_n^i(Y^1)|_{Y^1 = X^j} + DJ_n^2(Y^2)|_{Y^2 = X^j + K_1} \geq 0 = \partial_x V_n^{i-1}(\tilde{X}) \quad (Eq2)
\]

\[
\partial_x V_n^i(\tilde{X}) = DJ_n^i(Y^1)|_{Y^1 = X^j} + DJ_n^2(Y^2)|_{Y^2 = X^j + K_1} \\
\geq DJ_n^i(Y^1)|_{Y^1 = X^j} + DJ_n^2(Y^2)|_{Y^2 = X^j + K_1} = \partial_x V_n^{i-1}(\tilde{X}) \quad (Eq3)
\]

\[
\partial_x V_n^i(\tilde{X}) = DJ_n^i(Y^1)|_{Y^1 = X^j} \geq DJ_n^{i-1}(Y^1)|_{Y^1 = X^j} \\
\geq DJ_n^i(Y^1)|_{Y^1 = X^j} + DJ_n^{i-1}(Y^2)|_{Y^2 = X^j + K_1} = \partial_x V_n^{i-1}(\tilde{X}) \quad (Eq4)
\]

\[
\partial_x V_n^i(\tilde{X}) = DJ_n^i(Y^1)|_{Y^1 = X^j} \geq DJ_n^{i-1}(Y^1)|_{Y^1 = X^j} = \partial_x V_n^{i-1}(\tilde{X}) \quad (Eq5)
\]

\[
\partial_x V_n^i(\tilde{X}) = 0 \geq \partial_x V_n^{i-1}(\tilde{X}) \quad (Eq6)
\]

\[
\partial_x V_n^i(\tilde{X}) = DJ_n^i(Y^1)|_{Y^1 = X^j} \geq 0 = \partial_x V_n^{i-1}(\tilde{X}) \quad (Eq7)
\]

While the comparisons are fairly simple, the complicating factor is that different constraints (and corresponding different formula) describe equilibria in various regions. Figure 6 shows why various comparisons are needed and also gives intuition why they hold. It illustrates the dominance of derivatives for adjacent periods, but also shows why the ordering of base-stock levels is crucial. Consider case (A) and its two subcases: (I) \(z_{n-1}^1 < z_n^2 - K_1\) and (II) \(z_n^2 - K_1 \leq z_{n-1}^1\). Given the needed ordering of thresholds, all of the comparisons are straightforward. For example in Figure 6, derivatives are 0 below \(\min(z_{n-1}^1, z_n^1)\), in period \(n\) the derivative is positive between \(z_n^1\) and \(z_{n-1}^1\), while still 0 in period \(n - 1\). The ordering between \(\min(z_{n-1}^1, z_n^1)\) and the next threshold follows
from the inductional step, etc. Formally, we can relate each case and subcase to the reference inequalities (Eq1)-(Eq7) above:

(i, j) = (1, 1): For subcase (I), $X^1 \leq z_n^1$ → (Eq1), $z_n^1 \leq X^1 \leq \min(z_{n-1}^1, z_{n-1}^2 - K_1)$ → (Eq2); for subcase (II), $z_{n-1}^2 - K_1 \leq X^1 \leq z_{n-1}^2 - K_1$ → (Eq7)

(i, j) = (1, 2): For subcases (I) and (II), same as for (i, j) = (1, 1) for $X^1 \leq z_{n-1}^2 - K_1$

(i, j) = (1, 2): For subcase (I), $X^2 \leq z_n^1$ → (Eq3), $z_n^1 \leq X^2 \leq z_{n-1}^1$ → (Eq6), $z_{n-1}^1 < X^2 \leq z_n^2$ → (Eq1), $z_n^2 \leq X^2 \leq z_{n-1}^2$ → (Eq7), $z_{n-1}^2 - K_1 \leq X^2$ → (Eq5); for subcase (II), $z_n^1 \leq X^2 \leq \min(z_{n-1}^1, z_{n-1}^2 - K_1, z_{n-1}^2)$ → (Eq6)

(i, j) = (2, 2): same as for (i, j) = (1, 2)

Now, consider case (B) and its two subcases: (I) $z_{n-1}^2 - K_1 \leq z_{n-1}^1 \leq z_{n-1}^1 < z_{n-1}^2$ and (II) $z_{n-1}^1 < z_{n-1}^2$ - $K_1$. If this is not the case, the analysis reduces to fewer, simpler cases.

(i, j) = (1, 1): For subcase (I), $X^1 \leq z_n^2 - K_1$ → (Eq1), $z_n^2 - K_1 \leq X^1 \leq z_{n-1}^2 - K_1$ → (Eq7), $z_{n-1}^2 - K_1 \leq X^1 \leq z_{n-1}^2$ → (Eq4)

(i, j) = (2, 1): For subcases (I) and (II), $X^1 \leq z_{n-1}^2 - K_1$ → (Eq1)

(i, j) = (1, 2): For subcase (I), $X^2 \leq z_n^2 - K_1$ → (Eq3), $z_n^2 - K_1 \leq X^2 \leq z_{n-1}^2 - K_1$ → (Eq6), $z_{n-1}^2 - K_1 \leq X^2 \leq z_n^2$ → (Eq1); for subcase (II), $z_n^2 - K_1 \leq X^2 \leq z_{n-1}^2$ → (Eq6), $z_{n-1}^2 \leq X^1 \leq z_n^2$ → (Eq1)

(i, j) = (2, 2): same as for (i, j) = (1, 2), $z_n^2 \leq X^2 \leq z_{n-1}^2$ → (Eq7), $z_{n-1}^2 \leq X^2$ → (Eq5)

Case (C) has similar logic. Across cases (A), (B), and (C), the following is true $\partial V^1_n(\hat{X}) \geq \partial J^1_{n-1}(\hat{X})$ for $i, j = 1, 2$, and limited for $(i, j) = (1, 2)$, $X^2 \leq z_n^2$; and $(i, j) = (2, 1)$, $X^1 \leq \min(z_n^2 - K_1, z_n^1)$. Since we are dealing with the territory below the base-stock levels, these dominance conditions also hold for $E[V(Y - D)]$. Multiplying both sides by $\beta$ and adding $L'$ to both sides also maintains these conditions, resulting in $\partial J^1_{n+1}(Y) \geq \partial J^1_{n}(Y)$ for $i, j = 1, 2$, and limited for $(i, j) = (1, 2), Y^2 \leq z_n^2$, and $(i, j) = (2, 1), Y^1 \leq \min(z_n^2 - K_1, z_n^1)$. Given the convexity shown for the necessary territory, this is sufficient for $(x)$, which in turn yields (ii). Notice that firm 1’s periodic cost (which has a minimum at $y_{n+1}^*$) is added to the discounted expected cost-to-go, $V^1_n$, that is constant in $X_1$ for $X^1 \leq z_n^1$, since the equilibrium is dependent only upon $X^2$ (and the equilibrium up-to levels for future periods is higher). Formally, $\partial V^1_n = 0$ for $X^1 \leq \min(z_n^1, z_{n-1}^2 - K_1)$ in cases (A) and (B) and for $X^1 \leq z_n^1$ for case (C). This is preserved under the operation $\beta E[V^1_n(Y - D)]$ and thus $z_{n+1}^1 = y_{n+1}^* = \arg \min_{Y^1} [L^1(Y) + \beta E[V^1_n(Y - D)]]$ yielding (xi).
• Independence of the best response outside the band:

We now consider the state space above the band, $Y^{2} \geq Y^{1} + K_{1}$. From (xii), $J_{n}^{1}$ is convex in $Y^{1}$, decreasing in $Y^{1} \leq z_{n}^{1}$ for $Y^{2} \leq z_{n}^{2} + 2K_{1}$, independent of $Y^{2}$ for $Y^{2} \leq z_{n-1}^{2}$, separable for $Y^{1} + K_{1} \leq Y^{2} \leq Y^{1} + 2K_{1}$, and $J_{n}^{1}(Y^{1}, Y^{1} + K_{1})$ is decreasing for $Y^{1} \leq z_{n}^{1}$. Thus, the response function $r_{n}^{1}$ is within the band (for $Y^{2} \leq z_{n}^{1} + K_{1}$ it properly describes the best response function when the supplier is not limited to the band).

From (xiii), for $z_{n}^{1} + K_{1} \leq X^{2} \leq z_{n}^{1} + 2K_{1}$, the retailer’s best response is independent of $Y^{2}$, $r_{n}^{1}(Y^{2}) = z_{n}^{1}$ (i.e., it will rise “vertically” in the slice $B^{K_{1}}$, above the band), and consequently $D_{n}^{1}(Y^{2}) = 0$ for $z_{n}^{1} + K_{1} \leq Y^{2} \leq z_{n}^{2} + 2K_{1}$.

Now consider $X \in B^{K_{1}}$. For notational efficiency, let $z = \min(z_{n}^{1} - K_{1}, z_{n}^{2} - 2K_{1})$. For $X^{1} \leq z$, $V_{n}^{1}(X) = J_{n}^{1}(X^{1} + K_{1}, X^{1} + 2K_{1})$, adopting the decreasing convexity from the upper edge of the band, $B_{n}$: for $z \leq X^{1} \leq z + K_{1}$ and $X^{2} \leq z + 2K_{1}$, $V_{n}^{1}(X) = J_{n}^{1}(z + K_{1}, z + 2K_{1})$; for $z + 2K_{1} \leq X^{2} \leq z_{n}^{1} + K_{1}$, $V_{n}^{1}(X) = J_{n}^{1}(X^{2} - K_{1}, X^{2})$; for $z_{n}^{1} - K_{1} \leq X^{1} \leq z_{n}^{1}$ and $z_{n}^{1} + K_{1} \leq X^{2} \leq z_{n}^{1} + 2K_{1}$, $V_{n}^{1}(X) = J_{n}^{1}(z_{n}^{1}, X^{2})$: for $X^{1} \geq z_{n}^{1}$, $V_{n}^{1}(X) = J_{n}^{1}(X^{1}, X^{2})$. The important element is that the solution follows the upper edge of the band up to $\min(z_{n}^{1}, z_{n}^{2} - K_{1})$ resulting in the retailer’s decreasing convexity in $X^{1}$ and a zero slope for both the retailer and supplier with respect to $X^{2}$ within $B^{K_{1}}$ for $X^{2} \leq z + K_{1}$.

Thus, from the assumed convexity of $J_{n}^{1}$ in $Y^{1}$, the separability of $J_{n}^{1}$ and $\partial_{2}J_{k+1}^{1} = 0$, we have $V_{n}^{1}$ is convex decreasing in $X^{1} \leq z$, independent of $X^{2}$ for $z \leq X^{1} \leq z_{n}^{1}$, increasing convex for $X^{1} \geq z_{n}^{1}$, and constant in $X^{2}$ for $X^{2} \leq z_{n}^{2}$. Thus, $V_{n}^{1}(X) = V_{n}^{1}(X^{1} + 1) + V_{n}^{2}(X^{2})$. These separability and convexity in $X^{1}$ properties are preserved for the operation $\beta E[V_{n}^{1}(Y^{1} - D)]$. Adding $L^{1}(Y^{1})$ will then yield $J_{n+1}^{1}(\tilde{Y})$ which inherits the convexity and separability properties. As the minimizer of $J_{n+1}^{1}(\tilde{Y})$ is $z_{n+1}^{1}$, independent of $Y^{2}$ for $z_{n+1}^{1} + K_{1} \leq Y^{2} \leq z_{n+1}^{1} + 2K_{1}$, the first claim in (xiii) follows. Consequently convexity and monotonicity of $J_{n+1}^{1}(\tilde{Y})$ for $Y_{1} \leq z_{n+1}^{1}$ follows. As, $V_{n}^{1}$ has zero slope in $X^{2}$ for $X^{2} \leq z + K_{1}$ and positive slope in $X^{2}$ for $X^{2} > z + K_{1}$, properties conveyed to $J_{n+1}^{1}$. Thus, all properties are inherited yielding (xii) for $X \in B^{K_{1}}$.

For $X^{2} \geq X^{1} + 2K_{1}$, from convexity of $J_{n}^{1}$ in $Y^{1}$ and the definition of $r_{n}$, we have $V_{n}^{1}(X) = J_{n}^{1}(X^{1} + K_{1}, X^{2})$. Consequently, $V_{n}^{1}$ is convex decreasing in $X^{1} \leq z_{n}^{1}$ and independent of $X^{2} \leq z_{n}^{2}$ and convexity in $X^{1}$ properties are preserved for the operation $\beta E[V_{n}^{1}(\tilde{Y} - D)]$. Since convexity holds at $Y^{2} = Y^{1} + 2K_{1}$, it holds across all $Y^{1}$. Also, the increasing convexity of $J_{n+1}^{1}$ for $Y^{1} \geq z_{n+1}^{1}$ implies $r_{n+1}^{1}(Y^{2}) \leq Y^{2} - 2K_{1}$ for $Y^{2} > z_{n+1}^{1} + 2K_{1}$, which completes (xiii). Consequently, the retailer’s best response will not return to the band $B$ for $Y^{2} > z_{n+1}^{1} + 2K_{1}$.

Induction Basis: The salvage value function for player 1 is applied in period 0, while the salvage value function for player 2 is applied in period 1. Mirroring the claims in Lemma A1, for any $\epsilon \in (0, (h_{1} + h_{2})/4)$, let $\tau_{0}$ be such that $L'(\tau_{0}) > h_{1} + h_{2} - \epsilon$, and $\gamma_{1}$ such that for all $\tau \geq \gamma_{1}$, we have $L'(\tau) - \int_{\gamma_{1}}^{\infty} L'_{1}(\tau + K_{1} - D)f(D)dD < \epsilon$. We set (a) $\gamma_{1} = \max(y_{mg}, \tau_{0}, \gamma_{1}) + E[D]$ and $\lambda_{1} = (h_{1} + h_{2})/(4\beta E[D])$, (b) $\gamma_{2} = z_{1}^{1}$, and (c) $\lambda_{2} \geq \partial_{2}P_{E[D]}(D > \tau_{0} + 1)$. This will guarantee the ordering of derivatives below.

Period 1: We now allow $k$ to assume values of 0 and 1, corresponding to periods 1 and 2. In Period 1 we have:

$$J_{1}^{1}(\tilde{Y}) = E[(h_{1} + h_{2})(Y^{1} - D)^{+} + p_{1}(D - Y^{1})^{+}] + \beta\lambda_{1}E[(Y^{1} - D - \gamma_{1})^{2}]$$

$$J_{2}^{1}(\tilde{Y}) = h_{2}(Y^{2} - Y^{1}) + p_{2}E[(D - Y^{1})^{+}]$$

Clearly, $J_{1}^{1}(\tilde{Y}) = J_{1}^{1}(Y^{1}) + J_{1}^{2}(Y^{2})$, so $J_{1}^{2}(Y^{2}) = 0$, and $J_{2}^{1}(\tilde{Y}) = J_{2}^{1}(Y^{1}) + J_{2}^{2}(Y^{2})$, so $J_{2}^{1}(Y^{2}) = -h_{2}Y^{1} + p_{2}E[(D - Y^{1})^{+}]$ and $J_{2}^{2}(Y^{2}) = h_{2}Y^{2}$. Since $J_{1}^{1}$ are continuous in $\tilde{Y}$, convex in $Y^{1}$, and the action set, $A(X)$, is nonempty compact convex subset of a Euclidean space (as in the inducational

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3This case exists only when $z_{n}^{2} - K_{1} \leq z_{n}^{1}$. Moreover, this is the distorting case where the slope of the retailer’s best response for $X^{2} \geq X^{1} + 2K_{1}$ may be increased from zero over multiple periods.
step), from Theorem 1 there exists at least one pure-strategy Nash equilibrium, which occurs at the intersection of best-reply functions. Due to the strict convexity of $J^1_1$ with respect to $Y^1$ and the strict monotonicity of $J^2_2$, the equilibrium is unique in period 1.

Denote $z^1_i := \arg \min_{Y^1} J^1_1(Y^1)$ and $z^2_i := \arg \min_{Y^2} J^2_2(Y^2) = -\infty$. The equilibrium is:

$$Y^1 = z^1_i[[X^1, X^2]] \text{ and } Y^2 = X^2$$

yielding (i) for $n = 1$. Based on Karush's (1959) lemma $V^1_1(\tilde{X}) = V^1_1(X^1) + V^2_1(X^2)$ since $J^1_1$ is separable for $i = 1, 2$, establishing (ii). For (iv)-(vii), it is sufficient to show the desired properties for $V^1_1$. Note that Lemma 1 permits us to restrict attention to $\tilde{X} \in B$ only. From the definitions of $J^1_1$ (see above), the period 1 equilibrium, and the separability, we have that $V^1_1$ and $V^2_1$ are flat in $X^1 \leq z^1_1$ and convex increasing and convex decreasing, respectively, in $X^1 \geq z^1_1$ yielding (iv) and (v). $V^1_1(\tilde{X}) = J^1_1(X^2, X^2) = J^{11}_1(X^2)$ is convex decreasing in $X^2$ for $X^2 \leq z^1_1$ since $J^{11}_1$ is convex and minimized at $z^1_1$. $V^2_1(\tilde{X}) = J^2_1(\tilde{X}^2, \tilde{X}^2) = p^2_2E[(D - \tilde{X}^2)^+]$ is convex decreasing in $X^2$ for $X^2 \leq z^1_1$. $V^1_1(\tilde{X}) = J^1_1(\max(z^1_i, X^1), X^2) = J^{11}_1(\max(z^1_i, X^1)) + J^{22}_1(X^2)$ for $X^2 \geq z^1_1$.

Thus, $V^1_1(\tilde{X}) = J^{11}_1(\max(z^1_i, X^1))$ so $\partial_0 V^1_1 = 0$ and $V^2_1(\tilde{X}) = J^{12}_1(\max(z^1_i, X^1)) + J^{22}_1(X^2) = J^{12}_1(\max(z^1_i, X^1)) + h^2_2X^2$ so $\partial_0 V^2_1 = h^2_2 > 0$ for $X^2 \geq z^1_1$. This means that $V^1_1$ is convex in $X^1$ and non-increasing in $X^{-1}$ yielding (v) and (vi). While in any period $n \geq 2$, the analysis within band $B$ and outside this band will differ, for period 1 the slope (derivative) of $V^1_1$ with respect to $X^1$ is constant in $X^2$. Thus, the derived best reply function in period 2, $r^2_i$, will be completely vertical along the $Y^2$ dimension, thus justifying (xiii). Likewise, (xii) is justified since $J^1_1$ is convex in $Y^1$ around $z^1_1$ and $J^2_2$ is increasing in $Y^2$.

**Period 2**: From the definition of the salvage value functions above, $\partial_1 S^1_2(\tilde{X}) = 0$ and

$$\partial_2 S^1_2(\tilde{X}) = \begin{cases} -\lambda_2 & X^2 \leq \gamma_2 \\ 0 & \gamma_2 \leq X^2 \end{cases}$$

so $S^1_2$ is convex non-increasing in $X^2$ and independent of $X^1$. Thus, $S^1_2 + V^2_1$ is convex in $X^2$ and non-increasing in $X^1$. Consequently, $\beta E[S^1_2(\tilde{Y} - D)] + \beta E[V^2_1(\tilde{Y} - D)] + L^2(\tilde{Y}) =: J^2_2(\tilde{Y})$ has the same properties, yielding (vi) and (vii). Condition (3) in the main paper ensures a finite minimizer with respect to $Y^2$. Likewise, $V^1_1(\tilde{X})$ is convex in $Y^1$ and convex non-increasing in $Y^2$, implying that $J^2_2(\tilde{Y}) := L^1(\tilde{Y}) + \beta E[V^1_1(\tilde{Y} - D)]$ has these same properties, yielding (iv) and (v). This establishes the basis convexity conditions. Specifically, $\partial_1 V^1_1 = 0$ for $Y^1 \leq z^1_1$ and convex non-decreasing for $Y^1 > z^1_1$, which remains true for $\beta E[V^1_1(\tilde{Y} - D)]$. Thus, $z^2_2 = \arg \min_{Y^2} J^2_2 = g^2_{my}$, establishing (xii). $J^2_2$ is clearly separable since $L^i$ is separable.

Condition (a) is sufficient to establish that $z^i_2 \leq z^i_1$, partly showing (iii), since the upper edge of the band for $Y^1 \leq z^1_1$ mimicks the single dimensional state in the single echelon capacitated problem described in Lemma A1, under identical salvage value conditions. This will then establish the slope of $J^2_2(Y^1, Y^1 + K_1)$ at $(z^1_1, z^1_1 + K_1)$ will be non-negative, sufficient to establish (ix) for the retailer, and combined with the convexity results, sufficient to show (x) for the retailer. It is straightforward to see the equilibrium solution is $Y^2_2 = \max(z_2^i, \min(z^i_2, z^i_2 - K_1))[[X^1, X^2]]$ and $Y^2_2 = z^2_2[[X^2, Y^2_1 + K_1]]$.

To establish the basis for $z^2_{n+1} \leq z^2_n$, however, we need to consider an additional period since $z^2_n = -\infty$. Clearly, there exists the freedom to choose values of $\gamma_1$ and $\gamma_2$ to achieve $z^2_n \leq z^2_1$, which we do (partially establishing (iii)).

We consider the same three cases: (A) $z^1_2 \leq z^2_2 - K_1$, (B) $z^1_2 < z^2_2 - K_1 \leq z^2_1$, and (C) $z^2_2 - K_1 \leq z^2_1$. Consider first (A). Using the derived equilibrium solution $(Y^1, Y^2)$ above, we
have
\[
\partial_2 V_2^2(\tilde{X}) = \partial_1 J_2^2(\tilde{Y})|_{Y^1 = X^2} + \partial_2 J_2^2(\tilde{Y})|_{Y^2 = X^2 + K_1}. 
\] (Eq 8)
For \(X^2 \leq z_1^1\), \(\partial_2 V_1^2(\tilde{X}) = D[p_2 E[(D - X^2)^+]]) < 0\); for \(X^2 \geq z_1^1\), \(\partial_2 V_2^2(\tilde{X}) = h_2 > 0\). Thus, \(V_2^2\) has a finite minimizer in \(X^2\) and \(V_1^2(X) + S_1^2(X)\) has a finite minimizing point with respect to \(X^2\) due to the functional shape of \(S_1^2\). So \(J_2^2\) is convex in \(Y^2\) around \(z_2^2 \leq z_1^1\).

\[
J_2^2(\tilde{Y}) = L^2(\tilde{Y}) + \beta E[V_2^2(\tilde{Y} - \tilde{D})] + \beta E[S_1^2(\tilde{Y} - \tilde{D})]
\]

\[
= h_2(Y^2 - Y^1) + p_2 E[(D - Y^2)^+] + \beta p_2 E[(D + D - Y^2)^+] + \beta \lambda_2 E[(D + \gamma_2 - Y^2)^+].
\]
Thus, \(\partial_1 J_2^2 = \partial_1 J_2^2\), partly showing (x). For \(X^2 \leq z_1^1\), \(\partial_2 V_2^2(\tilde{X}) = D[p_2 E[(D - Y^1)^+]|_{Y^1 = X^2} + \partial_2 \beta E[V_2^1(\tilde{Y} - \tilde{D})]|_{Y^2 = X^2 + K_1} + \partial_2 \beta E[S_1^2(\tilde{Y} - \tilde{D})]|_{Y^2 = X^2 + K_1} = -p_2 \Pr(\tilde{D} > X^2) - \beta p_2 \Pr(D > X^2 + K_1 - D) - \beta \lambda_2\), and for \(X^2 \leq \gamma_2\), \(\partial_2 V_1^2(X) + S_1^2(X) = D[p_2 E[(D - Y^1)^+]|_{Y^1 = X^2} - \lambda_2 = -p_2 \Pr(D > X^2) - \lambda_2\). Since \(z_1^2 \leq z_1^1 = \gamma_2\), from condition (b) and the expression for \(X^2 \leq z_1^2\),

\[
\partial_2 V_2^2(\tilde{X}) = -p_2 \Pr(D > X^2) - \beta p_2 \Pr(D > X^2 + K_1 - D) - \beta \lambda_2
\]

\[
\geq -p_2 \Pr(D > X^2) - \lambda_2 = \partial_2 V_1^2(\tilde{X}) + S_1^2(\tilde{X})
\]
since we assume condition (c) of the theorem statement holds. To see condition (c) is sufficient,

\[
\lambda_2 \geq \frac{\beta p_2 \Pr(D + D > K_1)}{1 - \beta} \geq \frac{\beta p_2 \Pr(D + D > X^2 + K_1)}{1 - \beta}.
\]
In case (B), in period 2, for \(X^2 \leq z_2^2 - K_1\), \(\partial_2 V_2^2(\tilde{X}) = D[p_2 E[(D - Y^1)^+]|_{Y^1 = X^2} + \partial_2 \beta E[V_2^2(\tilde{Y} - \tilde{D})]|_{Y^2 = X^2 + K_1} + \partial_2 \beta E[S_1^2(\tilde{Y} - \tilde{D})]|_{Y^2 = X^2 + K_1} = -p_2 \Pr(D > X^2) - \beta p_2 \Pr(D > X^2 + K_1 - D) - \beta \lambda_2 - \partial_2 V_2^2(\tilde{X}) + S_1^2(\tilde{X})\) and for \(z_2^2 - K_1 \leq X^2 \leq z_2^2\), \(\partial_2 V_2^2(\tilde{X}) = 0 \geq -p_2 \Pr(D > X^2) - \lambda_2 = \partial_2 V_2^2(\tilde{X}) + S_1^2(\tilde{X})\).

For (A), (B), and (C), for \(X^2 \leq \max(z_1^1, \min(z_2^1, z_2^2 - K_1))\)

\[
\partial_2 V_2^2(\tilde{X}) \geq \partial_2 [V_2^2(\tilde{X}) + S_1^2(\tilde{X})]
\]

\[
\partial_2 E[V_2^2(\tilde{Y} - \tilde{D})] \geq \partial_2 E[V_2^1(\tilde{Y} - \tilde{D})] + \partial_2 E[S_1^2(\tilde{Y} - \tilde{D})]
\]

\[
\partial_2 [L^2(\tilde{Y}) + \beta E[V_2^2(\tilde{Y} - \tilde{D})] \geq \partial_2 [L^2(\tilde{Y}) + \beta E[V_1^2(\tilde{Y} - \tilde{D})] + \beta E[S_1^2(\tilde{Y} - \tilde{D})]]
\]

\[
\partial_2 J_2^2(\tilde{Y}) \geq \partial_2 J_2^2(\tilde{Y})\text{ for } Y^2 \leq z_2^2.
\]
For \(\min(z_2^1, z_2^2 - K_1) \leq X^2 \leq z_2^2\),

\[
\partial_2 V_2^2(\tilde{X}) = 0 \geq \partial_2 V_2^2(\tilde{X}) + \partial_2 S_1^2(\tilde{X})
\]
and following the same steps, we achieve

\[
\partial_2 J_3^2(\tilde{Y}) \geq \partial_2 J_2^2(\tilde{Y})
\]
showing the final part of (x), implying \(z_3^2 \leq z_2^2\), yielding (iii), due to the convexity already shown. The analysis deriving the properties for firm 1’s best reply function outside the band is very similar to that in the induction step, but slightly simpler since \(r_2^1(Y^2)\) is completely straight for values of \(Y^2 \geq Y^1 + K_1\), above the band \(B\). This completes the basis. ■
Theorem 3 If $p_2 \leq h_2(1-\beta)/\beta$, for each starting inventory $\tilde{X} \in B$, $S_0^1(\tilde{X}) = 0$ and $S_0^2(\tilde{X}) = 0$, there exists a unique pure-strategy Nash equilibrium where the retailer orders up to a myopic base-stock level, $y_{my}^*$, if possible, and the supplier orders no goods at all. Also, $V_n^i(\tilde{X}) = V_n^{1i}(X^1) + V_n^{2i}(X^2)$ for $i = 1, 2$, and $n > 0$.

Proof In period 1, $J_1^1(\tilde{Y}) = J_1^{1i}(Y^1) + J_1^{2i}(Y^2)$, $\partial_i \partial_j J_1^1 \geq 0$ for $i, j = 1, 2$, $y_{my}^* = \arg \min_{y_{my}^*} J_1^1$, $\partial_2 J_1^1 = 0$, $\partial_1 J_1^2 = -h_2-p_2 \Pr(D > Y^1) < 0$, and $\partial_i J_2^1 = h_2 > 0$. The constrained best-reply functions are $r_1^2(Y^2) = y_{my}^*[X^1, X^2]$ and $r_2^1(Y^1) = X^2$, creating a unique equilibrium. Thus, $V_1^i(\tilde{X}) = J_1^1(y_{my}^*[X^1, X^2], X^2)$.

Assume: $J_{n-1}^i(\tilde{Y}) = J_{n-1}^{1i}(Y^1) + J_{n-1}^{2i}(Y^2)$ for $i = 1, 2$, $\partial_i \partial_j J_{n-1}^1 \geq 0$ for $i = j$ and $i, j = 1, 2$, $y_{my}^* = \arg \min_{y_{my}^*} J_{n-1}^1$, $\partial_2 J_{n-1}^1 \leq 0$, $\partial_1 J_{n-1}^2 \leq 0$, $\partial_1 \partial_2 J_{n-1}^1 \geq 0$ for $Y^1 \leq y_{my}^*$, $\partial_2 J_{n-1}^2 \geq h_2-p_2 \sum_{i=1}^{n-2} \beta^i$. The constrained best-reply functions are $r_{n-1}^2(Y^2) = y_{my}^*[X^1, X^2]$ and $r_{n-1}^1(Y^1) = X^2$, creating a unique equilibrium. Thus, $V_{n-1}^i(\tilde{X}) = J_{n-1}^i(y_{my}^*[X^1, X^2], X^2)$ and is separable since $J_{n-1}^i$ is separable. It is clear $\partial_1 V_{n-1}^i = 0$ for $X^1 \leq y_{my}^*$ for $i = 1, 2$, $\partial_1 V_{n-1}^i \geq 0$, $\partial_1 \partial_2 V_{n-1} \leq 0$, $\partial_2 \partial_2 V_{n-1} \geq 0$, $\partial_2 V_{n-1} \leq 0$ for $Y^1 > y_{my}^*$, $\partial_2 V_{n-1} \geq -p_2 \Pr(D > Y^2) - p_2 \sum_{i=1}^{n-2} \beta^i \geq -p_2 \sum_{i=1}^{n-2} \beta^i$, $\partial_2 \partial_2 V_{n-1} \geq 0$. These relationships will be preserved for $E[V_{n-1}(\tilde{Y} - D)]$ and specifically $\partial_2 E[V_{n-1}^2] \geq -p_2 \sum_{i=0}^{n-2} \beta^i$. Multiplying by $\beta$ and adding the periodic cost generates $J_n$, which have all the requisite properties and specifically, $\partial_2 J_n = h_2 + \beta \partial_2 E[V_{n-1}^2] \geq h_2 - p_2 \sum_{i=1}^{n-1} \beta^i \geq h_2 - \frac{p_2}{1-\beta} \geq 0$ from the condition in the theorem. 

Theorem 4 The equilibrium up-to levels are non-increasing when $K_1, h_1$, or $h_2$ increase or when $p_1$ or $p_2$ decrease.

Proof We partition the analysis between that for the economic parameters $(h_1, h_2, p_1, p_2)$ and the capacity $(K_1)$, primarily in the basis for the induction. Firstly, consider the economic parameters and the two conditions: (a) $p_2 > h_2(1-\beta)/\beta$ and (b) $p_2 \leq h_2(1-\beta)/\beta$. Let us consider changes in the cost parameters, one at a time, but which do not change the status of condition of (a) or (b) for a particular instance of the model (we examine the situation where this is not the case later). Let us consider models satisfying condition (b) first. If condition (b) is maintained for the values described, then the results of Theorem 3 are true. Theorem 3 shows that player 2 orders nothing ($z^2_0 = -\infty$) and player 1 orders up to $y_{my}^*$, if possible. Since $y_{my}^* := \arg \min_{y_{my}^*} L^1(Y)$, increasing $h_1$ or $h_2$ or decreasing $p_1$ results in a non-increase in $y_{my}^*$, while $p_2$ has no effect upon $y_{my}^*$. We will make use of the convexity and separability results of Theorem 2 and the best-response construction results of Lemma 2.

Basis for Induction

Let us now consider models satisfying condition (a). Consider the economic parameters $(h_1, h_2, p_1, p_2)$ first. (Salvage value functions are the same for the different values of these parameters.) We consider the case for $h_1$ in depth and describe briefly how the other three cases differ slightly:

$$J_1^1(\tilde{Y}) = E[(h_1 + h_2)(Y^1 - D)^+ + p_1(D - Y^1)^+] + \beta \lambda_1 E[(Y^1 - D - \gamma_1)^2]$$
$$J_1^2(\tilde{Y}) = h_2(Y^2 - Y^1) + p_2 E[(D - Y^1)^+].$$

It is clear that $\partial[\partial_1 J_1^1(\tilde{Y})]/\partial h_1 = \Pr(D \leq Y^1) \geq 0$ for all $Y^1$. In addition, $\partial[\partial_2 J_1^2(\tilde{Y})]/\partial h_1 = 0$, $\partial[\partial_2 J_1^1(\tilde{Y})]/\partial h_1 = 0$, and $\partial[\partial_1 J_1^2(\tilde{Y})]/\partial h_1 = 0$. Since $J_1^1(\tilde{Y})$ is independent of $Y^2$, $z_1^1 = \arg \min_{y_{my}^*} J_1^1(\tilde{Y})$, $\partial[\partial_1 J_1^1(\tilde{Y})]/\partial h_1 \geq 0$ and the convexity of $J_1^1$ results in $\partial z_1^1/\partial h_1 \leq 0$.

Now consider the basis for the capacity. Clearly, $\partial[\partial_1 J_1^1(\tilde{Y})]/\partial K_1 = \partial[\partial_2 J_1^2(\tilde{Y})]/\partial K_1 = \partial[\partial_2 J_1^2(\tilde{Y})]/\partial K_1 = \partial[\partial_1 J_1^2(\tilde{Y})]/\partial K_1 = 0$, implying $\partial z_1^1/\partial K_1 = 0$. 

8
Induction Step

Now for period \(n\), \(\partial[\partial J^1_n(\hat{Y})]/\partial h_1 \geq 0, \partial[\partial J^2_n(\hat{Y})]/\partial h_1 \geq 0\) for all \(Y^2 \leq z^2_n\), \(\partial[\partial J^0_n(\hat{Y})]/\partial h_1 \geq 0\), and \(\partial[\partial J^3_n(\hat{Y})]/\partial h_1 \geq 0\) for all \(Y^1 \leq \min(z^2_n, -K_1, z^1_n)\). (The analysis will be performed for \(h_1\) but is very similar for the other economic parameters and capacity.)

From the convexity and separability of \(J^2_n\), and \(\partial[\partial J^2_n(\hat{Y})]/\partial h_1 \geq 0\) immediately implies \(\partial z^2_n/\partial h_1 \leq 0\) and \(\partial r^2_n/\partial h_1 \leq 0\), which has the structure described in Lemma 2. Likewise, from the convexity and separability of \(J^3_n\), \(\partial[\partial J^0_n(\hat{Y})]/\partial h_1 \geq 0, \partial[\partial J^1_n(\hat{Y})]/\partial h_1 \geq 0\), then \(\partial z^1_n/\partial h_1 \leq 0\), \(\partial r^1_n/\partial h_1 \leq 0\). For the cases of increasing \(J^1_n\) to \(K_1\) and \(\partial J^3_n\) to \(Y^1\), the steps of analysis are very similar to that of \(h_1\). Now, \(\partial[D_J^1_n(Y^1, Y^1+K_1)]/\partial K_1 = \partial[\partial J^1_n(\hat{Y})]/\partial K_1+\partial[\partial J^3_n(\hat{Y})]/\partial K_1 = \partial[\partial J^1_n(\hat{Y})]/\partial K_1+\partial[\partial J^3_n(\hat{Y})]/\partial K_1=0\).

Thus, \(\partial z^1_n/\partial h_1 = 0\). From Theorem 2, the convexity of \(J^3_n\) is mapped to \(V^4_n\) via the equilibrium solution: \(\partial[\partial J^1_n(\hat{Y})]/\partial h_1 \geq 0, \partial[\partial J^3_n(\hat{Y})]/\partial h_1 \geq 0\) for all \(Y^2 \leq z^2_n, \partial[\partial J^3_n(\hat{Y})]/\partial h_1 \geq 0, \partial[\partial J^3_n(\hat{Y})]/\partial h_1 \geq 0\) for all \(Y^1 \leq \min(z^2_n, -K_1, z^1_n)\).

What about a change in the cost parameters \(p_2\) and \(h_2\) whereby there is a move from condition (a) to condition (b)? (The discussion will be identical for the converse case.)

References